

HEAVY-QUARK SYMMETRY AND SKYRMIONS

Dong-Pil Min

*Department of Physics and Center for Theoretical Physics,
Seoul National University, Seoul 151-742, Korea*

Yongseok Oh

Department of Physics, National Taiwan University, Taipei, Taiwan 10764, R.O.C.

Byung-Yoon Park

Department of Physics, Chungnam National University, Daejeon 305-764, Korea

and

Mannque Rho

Service de Physique Théorique, C. E. Saclay, 91191 Gif-sur-Yvette Cedex, France

ABSTRACT

We review recent development on combining heavy-quark symmetry and chiral symmetry in the skyrmion structure of the baryons containing one or more heavy quarks, c (charmed) and b (bottom). We describe two approaches: One going from the chiral symmetry regime of light quarks to the heavy-quark symmetry regime which will be referred to as “bottom-up” approach and the other going down from the heavy-quark limit to the realistic finite-mass regime which will be referred to as “top-down.” A possible hidden connection between the two symmetry limits is suggested. This review is based largely on the work done – some unpublished – by the authors since several years.

(To be published in Int. J. Mod. Phys. E)

Contents

1	Introduction	1
2	Effective Lagrangian	6
2.1	Chiral Symmetry	6
2.2	Heavy Quark Symmetry	11
3	Skyrme Model and Hidden Symmetry: “Bottom-Up” Approach	15
4	Effective Field Theory for Heavy Mesons	24
4.1	Heavy-Quark Effective Theory	24
4.2	Heavy Meson Theory	27
5	Heavy Baryons as Skyrmions	33
5.1	Heavy-Meson-Soliton Bound State	33
5.2	Collective Coordinate Quantization	41
5.3	Alternative Approach	47
6	Further Developments	52
6.1	Light Vector Mesons	52
6.2	Finite Mass Corrections	57
7	Conclusions	67
	Acknowledgement	68
	References	68

1. Introduction

Hadrons containing a single heavy quark (Q) with its mass (m_Q) much greater than a typical scale of strong interaction (Λ_{QCD}) can be viewed as a freely propagating point-like color source dressed by light degrees of freedom, namely the chiral quarks and gluons (sometimes referred to as “brown muck”). In addition to the chiral symmetry governing the dynamics of the light quark system, such a system reveals a new spin and flavor symmetry,^{1,2,3,4} “heavy quark symmetry,” as the heavy quark mass goes to infinity. (For a review, see Refs. 5,6,7,8.) In this limit, the heavy quark spin \vec{S}_Q decouples from the rest of the strongly interacting light quark system, since its coupling is a relativistic effect of order $1/m_Q$. Furthermore, the dynamics of a heavy quark in QCD depends only on its velocity and is independent of its mass, *i.e.*, the flavor. Consider two hadrons A and B made of a single heavy quark of mass m_Q^A and m_Q^B , respectively, and light degrees of freedom of QCD. If the heavy quark masses are much greater than the scale of the QCD interactions, $m_Q^A, m_Q^B \gg \Lambda_{QCD}$, then in the rest frame of the heavy quark, how QCD distributes the light degrees of freedom around the static heavy quark is independent of the heavy flavor. By boosting, one can extend the heavy flavor symmetry to any heavy quarks of the same *velocity* (not the same momentum). This symmetry is analogous to the isotope effect in atomic physics where the electronic structure of an atom is independent of the number of neutrons in the nucleus and the hyperfine splitting is of order $\sim 1/Am_N$.

The heavy quark symmetry provides an enormous help in reducing the large number of independent parameters required to describe the low momentum properties of the strong interaction involving a heavy quark. One of the well-known examples is that all six form factors for $B \rightarrow De\bar{\nu}_e$ and $B \rightarrow D^*e\bar{\nu}_e$ semileptonic decays are described by a universal function of velocity transfer, the “Isgur-Wise” function.³

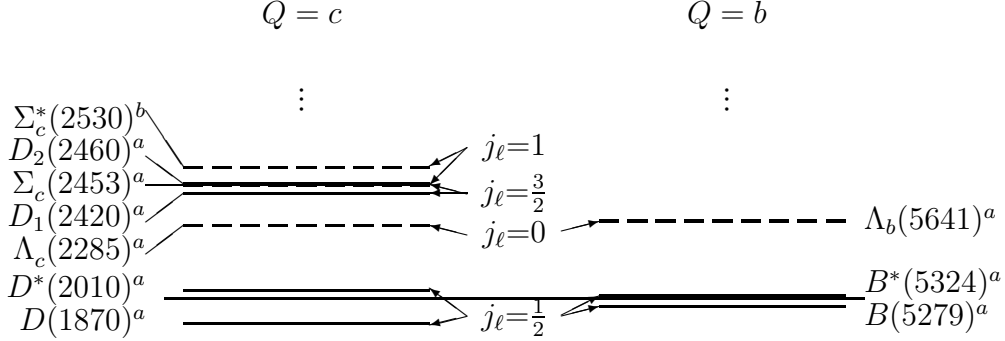
Another consequence of this heavy quark symmetry can be found in the spectra and strong decay widths of heavy hadrons.⁹ Due to the heavy quark spin decoupling, the total angular momentum of the light degrees of freedom $\vec{J}_\ell (\equiv \vec{J} - \vec{S}_Q$, with \vec{J} the spin of the hadron) is conserved and the corresponding quantum number j_ℓ can classify the hadrons and as a consequence, the hadrons come in degenerate doublets with total spin

$$j_\pm = j_\ell \pm \frac{1}{2}, \quad (1.1)$$

(unless $j_\ell = 0$) which are formed by combining the spin of the heavy quark with j_ℓ . Furthermore, the strong transitions between any two pairs of doubly degenerate states, occurring via the emission of light hadrons, are related simply by Clebsch-Gordan coefficients.⁹ For example, the ratio of the amplitudes for $\Sigma_Q \rightarrow \Lambda_Q \pi$ and $\Sigma_Q^* \rightarrow \Lambda_Q \pi$ will be unity. On the other hand, the heavy quark-flavor symmetry implies that if we line up the ground states by subtracting the mass of the heavy quark, then the spectra built on different heavy flavors should look the same. That is, the splittings are flavor independent.

Given in Fig. 1.1 are the experimentally observed mesons (solid lines) and baryons (dashed lines) with a single charm or bottom quark. The average masses of the ground state heavy meson doublets, $\bar{m}_D (\equiv \frac{3}{4}m_{D^*} + \frac{1}{4}m_D)$ and $\bar{m}_B (\equiv \frac{3}{4}m_{B^*} + \frac{1}{4}m_B)$, are lined up. The hadrons can be easily lumped into approximately degenerate doublets with

Figure 1.1 : Spectrum of hadrons containing a single heavy quark.
(^a) Ref. 10, (^b) Ref. 11)



$j_\pm = j_\ell \pm 1/2$. The D^*-D splitting of ~ 145 MeV is reduced to ~ 50 MeV in the B^*-B multiplet, which is consistent with the expected $1/m_Q$ behavior in the heavy-quark limit. In Fig. 1.1, one may infer the heavy quark flavor symmetry from the approximately same mass differences $m_{\Lambda_c} - \overline{m}_D$ and $m_{\Lambda_b} - \overline{m}_B$.

In the Skyrme model *à la* Callan and Klebanov (CK),^{12,13} heavy baryons can be described by bound states of soliton of the $SU(2)$ chiral Lagrangian and the heavy mesons containing the corresponding heavy quark. This picture which was originally put forward¹⁴ and shown to be successful^{15,16,17,18} for the strange baryons was first suggested by Rho, Riska and Scozzola¹⁹ to be applicable to baryons containing one or more charm (c) and bottom (b) quarks. The results on the mass spectra²⁰ and magnetic moments²¹ for charmed baryons were found to be strikingly close to the quark model description which is expected to work better as the heavy quark involved becomes heavier.

When the bound system of the soliton and heavy mesons is quantized to a heavy baryon of spin j and isospin i , the mass is given by the sum of the mass of the $SU(2)$ soliton, the meson energy and a hyperfine splitting. For the baryons with a single heavy flavor, the mass formula simply reads as^{12,14}

$$m_{(i,j)} = M_{sol} + \omega_B + \frac{1}{2\mathcal{I}} \{cj(j+1) + (1-c)i(i+1) + c(c-1)k(k+1)\}. \quad (1.2)$$

Here, M_{sol} and \mathcal{I} are the soliton mass and its moment of inertia with respect to the collective isospin rotation, respectively, and ω_B is the eigenenergy of the heavy meson bound state. The heavy meson bound state comes as an eigenstate of the grand spin operator $\vec{K} (\equiv \vec{I}_h + \vec{J}_h$ with \vec{J}_h and \vec{I}_h being the spin and isospin operator of heavy mesons, respectively), where k in Eq. (1.2) denotes the corresponding grand spin quantum number. The spin of the baryon j can take one of the values $|i-k|, \dots, i+k$. The

constant c is defined through

$$\langle k, k_3 | \vec{\Theta} | k, k_3 \rangle \equiv -c \langle k, k_3 | \vec{K} | k, k_3 \rangle, \quad (1.3)$$

in analogy to the Lande's g-factor in atomic physics. Here, $\vec{\Theta}$ is the meson field operator induced by the collective isospin rotation and it forms the first rank tensor in the space of the grand spin eigenstates $|k, k_3\rangle$. This "hyperfine constant" yields the hyperfine splittings between the heavy baryon masses and plays the role of an *order parameter* for the heavy quark symmetry.^{20,22} As the heavy meson masses increase, it decreases and becomes zero in the infinite heavy meson mass limit so that the heavy baryon masses do not depend explicitly on the *spin*.

In this review, we discuss the bound state approach of the Skyrme model to describe the heavy baryons. The plan of the review is as follows. In the remainder of this section we briefly discuss the successful and unsuccessful features of the straightforwardly extended CK model which does not respect the heavy quark symmetry explicitly. Section 2 is devoted to the construction of an effective meson Lagrangian which satisfies both the chiral symmetry and the heavy quark symmetry at infinite heavy quark mass limit. To construct more realistic models for finite heavy-quark mass, one may include $1/m_Q$ corrections to the effective theory. We refer to this approach as "top-down" approach. On the other hand, one may start from the CK model of the chiral symmetry regime and go to the heavy meson mass limit. We shall refer to this as the "bottom-up" approach. This will indicate whether and how the two theories can be related smoothly. Such an attempt is discussed in Sec. 3. In Sec. 4 we review the heavy meson effective theory in more detail. The spin and isospin operators of the heavy meson fields are obtained and the tensor representation of the heavy meson fields is introduced, which is very useful in calculating physical quantities at infinite mass limit. In Sec. 5 bound states of the soliton and heavy mesons are obtained and their quantization procedures are presented. The heavy baryon mass spectrum so obtained is based on the effective Lagrangian developed in the previous sections, which contains only the interactions of pions and heavy mesons. To include higher order interactions with pions one may include light vector meson degrees of freedom such as ω and ρ into the effective theory. Also for more realistic descriptions, we need to include the $1/m_Q$ corrections. Recent developments in these subjects are discussed in Sec. 6 and a summary is made in Sec. 7 with some conclusions.

The mass formula (1.2) can be generalized for the baryons with $n_i (i=1,2)$ mesons of species i (representing orbital state and flavor) with energy $\omega_{B,i}$, grand spin k_i and hyperfine constant c_i in an approximation analogous to the quasi-particle approximation in many-body physics with the residual interaction ignored. The generalized one

is in a slightly more complicated form:^{19,20}

$$\begin{aligned}
m_{(i,j;j_m)} = & M_{sol} + n_1\omega_{B,1} + n_2\omega_{B,2} \\
& + \frac{1}{2\mathcal{I}} \left\{ i(i+1) + (c_1 - c_2)[c_1j_1(j_1+1) - c_2j_2(j_2+1)] + c_1c_2j_m(j_m+1) \right. \\
& + [(c_1 + c_2)j_m(j_m+1) + (c_1 - c_2)(j_1(j_1+1) - j_2(j_2+1))] \\
& \left. \cdot \left[\frac{j(j+1) - j_m(j_m+1) - i(i+1)}{2j_m(j_m+1)} \right] \right\}, \tag{1.4}
\end{aligned}$$

where $j_i(i=1,2)$ is the total grand spin of the n_i mesons of species i and j_m is the total grand spin of the whole meson system. Due to the bosonic statistics involved, j_i is restricted to its maximum value that can be obtained by combining n_i mesons; that is, $j_i = n_i k_i$. However, since different orbital is populated and/or different flavor is considered, this does not hold for j_m ; here j_m can take one of the values $|j_1 - j_2|, \dots, j_1 + j_2$.

We show in Fig. 1.2 the resulting mass spectrum of the strange hyperons, charmed baryons and bottom baryons quoted from Refs. 19,23, which show a close resemblance to those of experiments¹⁰ and quark/bag models(QM: Ref. 24, BM: Ref. 25). The parameters used in obtaining the masses are presented in Table 1.1. Two quantities M_{sol} and $1/\mathcal{I}$ associated with the $SU(2)$ soliton are fitted to the nucleon and Δ masses by adjusting the parameters in the pionic sector. In **SM II** and **SM III**, ω_B 's and c 's are further fitted to the experimental masses of Λ , Σ , Λ_c and Σ_c . In the case of **SM I** and **SM IV**, the experimental meson masses m_K and m_B are used to obtain ω_B and c .

Although qualitatively successful, there have been a few problems in fine-tuning the Skyrme model to achieve a quantitative success. From a phenomenological point of view, the genuine mesonic parameters of the Skyrme Lagrangian leads to too deeply bound heavy baryons with somewhat large hyperfine splittings. One may improve the situation by modifying the symmetry breaking terms of the Lagrangian, for example, by incorporating the different values of the decay constants of the mesons of different flavor.^{20,21,26} The overbinding can be removed by increasing the ratio of the heavy meson decay constant to the pion decay constant but this leads to a larger hyperfine splitting,²⁷ which is certainly at odds with the heavy quark symmetry.

One can see that treating the heavy vector mesons in the traditional bound state approach¹⁸ cannot be compatible with the heavy quark symmetry, when straightforwardly extended. As far as the strange flavor is concerned, in analogy to the vector mesons ρ and π , the vector mesons K^* may be integrated out via *an ansatz* of the form

$$K_\mu^* = \frac{\sqrt{2}}{m_K^*} A_\mu K, \tag{1.5}$$

in favor of a combination of a background A_μ (see Sec. 2 for its detailed form) and the pseudoscalar meson field K . This approximation is valid, however, only when the vector meson masses are much heavier than those of the pseudoscalar mesons. Since the mass ratios of the pseudoscalar and vector mesons are not small enough for c- and b-quark mesons, as one can see in Table 1.2, the approximation of the type Eq. (1.5)

Figure 1.2 : The spectrum of (a) strange hyperons, (b) charmed baryons and (c) bottom baryons in the bound state approach.

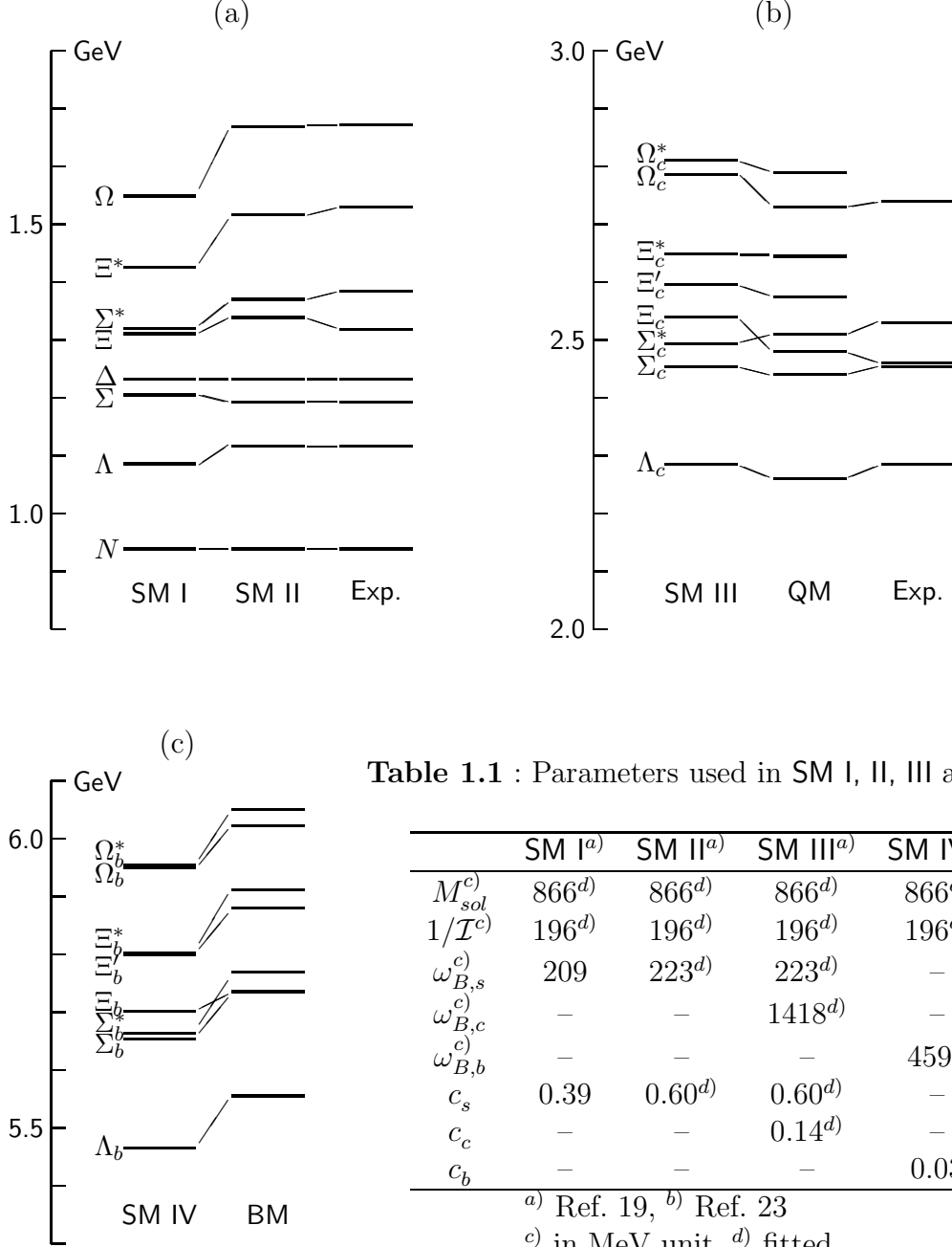


Table 1.2 : Masses of the pseudoscalar and vector mesons and their ratio.

	π, ρ	$Q = s$	$Q = c$	$Q = b$
Pseudoscalar Meson Mass (m_P)	138	498	1865	5278
Vector Meson Mass (m_{P^*})	770	892	2010	5325
Mass Ratio (m_P/m_{P^*})	0.18	0.56	0.93	0.99

can no longer be reliable. Furthermore, Eq. (1.5) suppresses the vector meson field by a factor inversely proportional to the vector meson mass, which is 0.56, 0.25 and 0.09, respectively, for the strange, charm and bottom sector when the explicit vector meson masses are substituted. It goes in the opposite direction to what we have expected of the heavy quark symmetry, according to which the role of the heavy vector mesons should become more important as the heavy vector meson mass becomes heavier and in the infinite mass limit, degenerate to that of the pseudoscalar meson.

This difficulty has been neatly resolved by Jenkins *et al.*,²⁸ whose work has led to a burst of publications in this field.^{29,30,31,32,33,34} The idea is to apply the bound-state approach to the heavy meson effective Lagrangian^{35,36} where the heavy quark symmetry *and* chiral symmetry are incorporated on the same footing. As shown in detail in Ref. 37, the heavy vector mesons play an essential role in giving correct hyperfine splitting in heavy baryons.

A remarkable new feature that emerges from the heavy-quark symmetry is that one can associate the structure of the hyperfine splitting with a non-abelian Berry potential.³⁴ In fact one can interpret certain baryonic excitations as following from induced gauge fields in appropriate flavor spaces and the vanishing of the hyperfine splitting can be identified with the vanishing of the Berry potential, in analogy to atomic and molecular systems. Readers interested in this matter are referred to Ref. 34 for details.

2. Effective Lagrangian

In this section, we discuss how one can construct a Lagrangian of heavy mesons interacting with light Goldstone bosons by incorporating simultaneously both the chiral symmetry and the heavy quark symmetry. We start with a familiar meson Lagrangian which yields Klein-Gordon equations for the spin-0 and spin-1 heavy meson fields. The elegant 4×4 ‘ $H(x)$ ’-matrix representation^{5,38,39} particularly designed for the heavy mesons — which may be however unfamiliar to some of the readers — will be discussed in Sec. 4.2.

2.1. Chiral Symmetry

The part of the QCD Lagrangian density involving the light degrees of freedom (light quarks and gluons) is

$$\mathcal{L} = -\frac{1}{4} \text{Tr} G^{\mu\nu} G_{\mu\nu} + \bar{q}(i\not{D}^c - m_q)q, \quad (2.1)$$

where the quark field $q = (u, d, \dots)^T$ is in the fundamental representation of both the color $SU(3)_c$ and the flavor $SU(N_f)$ and $G_{\mu\nu}^a$ is the gluon field strength with A_μ^a in the adjoint representation of $SU(3)_c$ ($a=1,2,\dots,8$). The covariant derivative^{#1} D_μ^c is

$$D_\mu^c = \partial_\mu + ig_s A_\mu^a \lambda^a, \quad (2.1a)$$

with the strong coupling constant g_s and m_q is the light quark mass matrix

$$m_q = \begin{pmatrix} m_u & 0 & \dots \\ 0 & m_d & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (2.1b)$$

For simplicity, we will work with two light flavors hereafter. The generalization to three flavors can be done straightforwardly.

Let $q_{R,L}$ be the right- and left-handed quark fields defined by

$$\begin{aligned} q_L &= \frac{1}{2}(1 - \gamma_5)q, \\ q_R &= \frac{1}{2}(1 + \gamma_5)q. \end{aligned} \quad (2.2)$$

Then the Lagrangian density can be expressed as

$$\mathcal{L} = -\frac{1}{4} \text{Tr} G^{\mu\nu} G_{\mu\nu} + \bar{q}_L i \not{D}^c q_L + \bar{q}_R i \not{D}^c q_R - (\bar{q}_L m_q q_R + \bar{q}_R m_q q_L).$$

Without the mass term, the Lagrangian is invariant under independent left- and right-transformations in the flavor space:

$$q_R \rightarrow q'_R = R q_R, \quad q_L \rightarrow q'_L = L q_L, \quad (2.3)$$

where R and L are arbitrary constant $SU(2)$ matrices. According to Noether's theorem, such an $SU(2)_L \times SU(2)_R$ chiral symmetry leads to the (classically) conserved left- and right-vector currents:

$$\partial^\mu (\bar{q}_R \frac{1}{2} \tau^i \gamma_\mu q_R) = 0, \quad \partial^\mu (\bar{q}_L \frac{1}{2} \tau^i \gamma_\mu q_L) = 0, \quad i = 1, 2, 3, \quad (2.4)$$

where the τ^i 's are the generators of the $SU(2)$ flavor in the fundamental representation. The corresponding $SU(2)$ conserved charges are

$$Q_R^i = \int d^3r q_R^\dagger \frac{1}{2} \tau^i q_R, \quad Q_L^i = \int d^3r q_L^\dagger \frac{1}{2} \tau^i q_L. \quad (2.5)$$

Chiral $SU(2)_L \times SU(2)_R$ symmetry is a symmetry of the Lagrangian density *but not of the vacuum*. The absence of the parity doublets in the physical spectrum suggests

^{#1}We will be working with various different forms of covariant derivatives associated with the chiral symmetry, the color gauge symmetry and the hidden gauge symmetry, and so on. In order to avoid confusion, we distinguish them by using different notations as D_μ , D_μ^c , \mathcal{D}_μ , etc.

that the $SU(2)_L \times SU(2)_R$ is *spontaneously* broken to $SU(2)_V$ leaving the Goldstone bosons, π^\pm and π^0 , associated with the broken generators. In other words, the QCD vacuum can have nonvanishing expectation value of the quark bilinear $\bar{q}_{aR}q_{bL}$

$$\langle 0 | \bar{q}_{aR}q_{bL} | 0 \rangle = v\delta_{ab}, \quad (2.6)$$

where $a, b(=u, d)$ are flavor indices. Under the $SU(2)_L \times SU(2)_R$ transformation, we see that

$$\langle 0 | \bar{q}_{aR}q_{bL} | 0 \rangle \rightarrow \langle 0 | \bar{q}_{cR}R_{ac}^*L_{bd}q_{dL} | 0 \rangle = R_{ca}^\dagger L_{bd} \langle 0 | \bar{q}_{cR}q_{dL} | 0 \rangle = R_{ca}^\dagger L_{bd} v\delta_{cd} = v(LR^\dagger)_{ba},$$

that is, the vacuum is invariant only when the transformation is restricted to the vector subgroup $SU(2)_V$ where $L = R$.

The quark mass term explicitly breaks the chiral $SU(2)_L \times SU(2)_R$ symmetry, giving the Goldstone bosons small masses. Quark masses can be treated as perturbation since the up and down current quark masses ($m_u \approx 5$ MeV and $m_d \approx 10$ MeV) are small compared with the QCD scale, $\Lambda_{QCD} \sim 200$ MeV, or the chiral symmetry scale $\Lambda_\chi \sim 1$ GeV.

The strong interactions at low energy can be described rigorously in terms of effective chiral Lagrangians involving Goldstone bosons. A powerful approach to this is chiral perturbation theory (χPT) which consists of systematic expansions in power of derivatives and the quark mass matrix. As is customarily done in the literature, we shall work with the $SU(2)_L \times SU(2)_R$ chiral symmetry realized *nonlinearly*. For this, we define the chiral field

$$U = e^{iM/f_\pi}, \quad (2.7)$$

where M is a 2×2 matrix for the triplet of Goldstone bosons (π^+, π^0 and π^-):

$$M = \vec{\tau} \cdot \vec{\pi} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}, \quad (2.7a)$$

and f_π is the pion decay constant

$$f_\pi = 93 \text{ MeV}. \quad (2.7b)$$

Under $SU(2)_L \times SU(2)_R$, U transforms

$$U \rightarrow U' = LUR^\dagger, \quad (2.7c)$$

where $L \in SU(2)_L$ and $R \in SU(2)_R$ are global transformation. Then the leading-order interactions of the Goldstone bosons are described by a single parameter, the pion decay constant,

$$\mathcal{L}_M = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \dots, \quad (2.8)$$

where terms with higher derivatives are abbreviated by the ellipsis. The factor $f_\pi^2/4$ gives a properly normalized kinetic terms for the pions:

$$\mathcal{L}_M = \frac{1}{2} \partial_\mu \pi^a \partial^\mu \pi^a + \dots$$

Witten⁴⁰ has observed that the non-linear σ -model Lagrangian (2.8) exhibits two discrete symmetries

$$(i) \ U(\vec{r}, t) \rightarrow U(-\vec{r}, t), \quad (ii) \ U(\vec{r}, t) \rightarrow U^\dagger(\vec{r}, t), \quad (2.9)$$

whereas QCD requires only the invariance of the Lagrangian under $U(\vec{r}, t) \rightarrow U^\dagger(-\vec{r}, t)$, that is, the combination of (i) and (ii). The second symmetry prohibits processes that involve n -pseudoscalar vertex where n is an odd number, *e.g.*, $K^+ K^- \rightarrow \pi^+ \pi^- \pi^0$ which is allowed by QCD. The latter is encoded in an anomalous term, known as “Wess-Zumino term”,⁴¹ which breaks (i) and (ii) separately but preserves their combination. The Wess-Zumino term cannot be written as a local Lagrangian density in $(3+1)$ dimensions but can be as a local action in five-dimensions,⁴⁰

$$\Gamma_{WZ} = -\frac{iN_c}{240\pi^2} \int_{M_5} d^5x \varepsilon^{\mu\nu\rho\sigma\lambda} \text{Tr}(U^\dagger \partial_\mu U U^\dagger \partial_\nu U U^\dagger \partial_\rho U U^\dagger \partial_\sigma U U^\dagger \partial_\lambda U), \quad (2.10)$$

where the integration is over a five-dimensional disk whose boundary is the ordinary space-time M_4 and U is extended so that $U(\vec{r}, t, s=0) = 1$ and $U(\vec{r}, t, s=1) = U(\vec{r}, t)$. This term is non-vanishing for $N_f \geq 3$. When the soliton is built in $SU(2)$ space, this term does not figure directly. However we shall be considering $(2+1)$ flavors where one flavor can be heavy in which case the dynamics can be influenced by the Wess-Zumino term as in the Callan-Klebanov model.

Consider a heavy meson containing a heavy quark Q and a light anti-quark \bar{q} . We will assume that the light anti-quark in the heavy meson forms a point-like object with the heavy quark, just providing them a color, flavor, spin and parity. Let P and P_μ^* be the operators that annihilate $J^P = 0^-$ and 1^- mesons, respectively. If, for example, the heavy quark is charmed, these fields form an $SU(2)$ anti-doublets:

$$P = (D^0, D^+) \quad \text{and} \quad P^* = (D^{*0}, D^{*+}). \quad (2.11)$$

Their conventional free field Lagrangian density is given by

$$\mathcal{L} = \partial_\mu P \partial^\mu P^\dagger - m_P^2 P P^\dagger - \frac{1}{2} P^{*\mu\nu} P_{\mu\nu}^{*\dagger} + m_{P^*}^2 P^{*\mu} P_\mu^{*\dagger}, \quad (2.12)$$

where $P_{\mu\nu}^* = \partial_\mu P_\nu^* - \partial_\nu P_\mu^*$ is the field strength tensor of the heavy vector meson fields P_μ^* , and m_P and m_{P^*} are the masses of the heavy pseudoscalar and vector mesons, respectively.

In order to construct a chirally invariant Lagrangian containing P , P_μ^* and their coupling to the Goldstone bosons, we need to assign to the heavy meson fields a transformation rule with respect to the full chiral symmetry group $SU(2)_L \times SU(2)_R$. There is a considerable freedom for doing this. The standard one is to introduce

$$\xi = U^{\frac{1}{2}}, \quad (2.13)$$

which transforms under $SU(2)_L \times SU(2)_R$ as

$$\xi \rightarrow \xi' = L \xi \vartheta^\dagger = \vartheta \xi R^\dagger, \quad (2.14)$$

where ϑ is a local unitary matrix depending on L , R and the Goldstone fields $M(x)$. From ξ we can construct an “induced” vector field V_μ and an “induced” axial vector field A_μ as

$$\begin{aligned} V_\mu &= \frac{1}{2}(\xi^\dagger \partial_\mu \xi + \xi \partial_\mu \xi^\dagger), \\ A_\mu &= \frac{i}{2}(\xi^\dagger \partial_\mu \xi - \xi \partial_\mu \xi^\dagger), \end{aligned} \quad (2.15)$$

which transform under chiral symmetry

$$\begin{aligned} V_\mu &\rightarrow V'_\mu = \vartheta V_\mu \vartheta^\dagger + \vartheta \partial_\mu \vartheta^\dagger, \\ A_\mu &\rightarrow A'_\mu = \vartheta A_\mu \vartheta^\dagger. \end{aligned} \quad (2.16)$$

The vector field V_μ behaves as a gauge field under the local chiral transformation, while the axial vector field transforms covariantly. Since the light quark doublet q transforms^{42,43}

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \rightarrow q' = \vartheta q,$$

the heavy meson anti-doublets, P and P^* whose quark contents are $Q\bar{q}$, transform as

$$P \rightarrow P' = P \vartheta^\dagger, \quad \text{and} \quad P^* \rightarrow P'^* = P^* \vartheta^\dagger. \quad (2.17)$$

We next define a covariant derivative in terms of the vector field V_μ

$$\begin{aligned} D_\mu P^\dagger &= (\partial_\mu + V_\mu) P^\dagger, \\ D_\mu P &\equiv (D_\mu P^\dagger)^\dagger = P (\overleftarrow{\partial}_\mu + V_\mu^\dagger), \end{aligned} \quad (2.18)$$

which transforms as

$$D_\mu P \rightarrow (D_\mu P)' = (D_\mu P) \theta^\dagger.$$

We can also construct similar equations for the vector meson fields P_μ^* .

Given the above definitions, it is an easy matter to write down the chirally invariant Lagrangian for P and P_μ^* with couplings to the Goldstone bosons. In terms of derivatives acting on the Goldstone boson fields, it has the form⁴⁴

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_M + D_\mu P D^\mu P^\dagger - m_P^2 P P^\dagger - \frac{1}{2} P^{*\mu\nu} P_{\mu\nu}^* + m_{P^*}^2 P^{*\mu} P_\mu^* \\ &\quad + f_Q (P A^\mu P_\mu^* + P_\mu^* A^\mu P) + \frac{1}{2} g_Q \varepsilon^{\mu\nu\lambda\rho} (P_{\mu\nu}^* A_\lambda P_\rho^* + P_\rho^* A_\lambda P_{\mu\nu}^*), \end{aligned} \quad (2.19)$$

where f_Q and g_Q are the $P^* P M$ and $P^* P^* M$ coupling constants, respectively, and the field strength tensor is

$$P_{\mu\nu}^* = D_\mu P_\nu^* - D_\nu P_\mu^*. \quad (2.20)$$

Note that the Lagrangian contains the $P^* P M$ and $P^* P^* M$ couplings but no PPM coupling that would violate parity invariance.

One can also work with heavy-meson fields defined differently. For instance, a suitable pair of heavy-meson fields are \hat{P} and \hat{P}_μ^* which transform as $(\bar{2}_L, 1_R)$ under $SU(2)_L \times SU(2)_R$; *viz.*,

$$\hat{P} \rightarrow \hat{P}' = \hat{P}L^\dagger, \quad \text{and} \quad \hat{P}_\mu^* \rightarrow \hat{P}_\mu^{*\prime} = \hat{P}_\mu^*L^\dagger, \quad (2.21)$$

with $L \in SU(2)_L$. These fields may appear simpler than those of Eq. (2.17), but their transformation under parity is a bit more complicated. Since parity interchanges left- and right-handed quark fields, the parity image of the hatted heavy meson fields must transform under $SU(2)_L \times SU(2)_R$ as $(1_L, \bar{2}_R)$; *i.e.*,

$$\mathcal{P}\hat{P}(\vec{r}, t)\mathcal{P}^{-1} = -\hat{P}(-\vec{r}, t)U(-\vec{r}, t) \xrightarrow{SU(2)_L \times SU(2)_R} -\hat{P}(-\vec{r}, t)U(-\vec{r}, t)R^\dagger, \quad (2.22)$$

and similarly for the vector meson fields. In contrast, the unhatted fields transform simply under parity (taking into account that the heavy meson has negative intrinsic parity)

$$\mathcal{P}P(\vec{r}, t)\mathcal{P}^{-1} = -P(-\vec{r}, t). \quad (2.23)$$

One can easily verify that the hatted fields are related to the unhatted ones by

$$\hat{P} = P\xi^\dagger \quad \text{and} \quad \hat{P}_\mu^* = P_\mu^*\xi^\dagger. \quad (2.24)$$

The Lagrangian Eq. (2.19) rewritten in terms of the hatted fields is

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_M + \hat{D}_\mu \hat{P} \hat{D}^\mu \hat{P}^\dagger - m_P^2 \hat{P} \hat{P}^\dagger - \frac{1}{2} \hat{P}^{*\mu\nu} \hat{P}_{\mu\nu}^\dagger + m_{P^*}^2 \hat{P}^{*\mu} \hat{P}_\mu^\dagger \\ & + \frac{i}{2} f_Q (\hat{P} U^\dagger \partial_\mu U \hat{P}_\mu^{*\dagger} + \hat{P}_\mu^* U^\dagger \partial_\mu U \hat{P}^\dagger) \\ & + \frac{i}{4} g_Q \varepsilon^{\mu\nu\lambda\rho} (\hat{P}_{\mu\nu}^* U^\dagger \partial_\lambda U \hat{P}_\rho^\dagger + \hat{P}_\rho^* U^\dagger \partial_\lambda U \hat{P}_{\mu\nu}^\dagger), \end{aligned} \quad (2.25)$$

where the “hatted” covariant derivatives denote

$$\begin{aligned} \hat{D}_\mu \hat{P} &= \partial_\mu \hat{P} + \hat{P} \frac{1}{2} \partial_\mu U^\dagger U, \\ \hat{D}_\mu \hat{P}_\nu^* &= \partial_\mu \hat{P}_\nu^* + \hat{P}_\nu^* \frac{1}{2} \partial_\mu U^\dagger U. \end{aligned} \quad (2.25a)$$

Note that the vector and axial vector currents of Eq. (2.19) do not appear in the Lagrangian (2.25). There appears only the term $U^\dagger \partial_\mu U$ which transforms as an $SU(2)_L$ triplet.

2.2. Heavy Quark Symmetry

In the heavy quark limit ($m_Q \gg \Lambda_{QCD}$), the subleading terms in $1/m_Q$ are suppressed and the heavy-quark symmetry becomes manifest in the Lagrangian: the heavy pseudoscalar and vector mesons become degenerate. This can be easily understood in quark models. In the constituent quark model, the P - P^* mass difference is due to the hyperfine splitting generated by the one-gluon exchange between constituent quarks,²⁴

$$m_{P^*} - m_P = \frac{\kappa}{m_Q m_q}, \quad (2.26)$$

where $m_Q(m_q)$ is the heavy (light)-quark mass and κ is a constant depending on the wavefunction of the hadron. If m_Q is sufficiently large compared with Λ_{QCD} , the wavefunction of the heavy-quark–light-anti-quark(s) bound state does not depend on the heavy-quark mass and the constant κ approaches an asymptotic value κ_∞ . Although experimental information is at the moment severely limited, the D^*-D mass difference¹⁰ of ~ 145 MeV and the B^*-B splitting⁴⁵ of ~ 50 MeV are consistent with the expected $1/m_Q$ behavior.

As the heavy-quark mass goes to infinity, the heavy-quark spin decouples from the rest of the strongly interacting light-quark system, namely the “brown muck,” since their coupling is a relativistic effect of order $1/m_Q$. This is referred to as “heavy-quark spin symmetry.” Furthermore, in that limit, the structure of the brown muck in the heavy meson is independent of the heavy-quark flavor. This is referred to as “heavy-quark flavor symmetry.” An analogy is found in the excitation spectrum and transition matrix element of a hydrogen-like atom, both of which are independent of the mass and spin of the nucleus. The heavy-quark spin symmetry relates two coupling constants f_Q and g_Q in the Lagrangian (2.19) and the heavy-quark flavor symmetry gives their dependence on the heavy-quark masses. The Lagrangian (2.19) and the axial current

$$A_\mu = -f_\pi \vec{\tau} \cdot \partial_\mu \vec{\pi} + \dots, \quad (2.27)$$

determine, at tree order, the matrix elements for the emission of a soft pion,

$$\begin{aligned} M(P^*(\varepsilon) \rightarrow P + \pi^a(q)) &= (\tfrac{1}{2}\phi^\dagger(P)\tau^a\phi(P^*))\frac{f_Q}{f_\pi}(\varepsilon \cdot q), \\ M(P^*(\varepsilon) \rightarrow P^*(\varepsilon') + \pi^a(q)) &= (\tfrac{1}{2}\phi^\dagger(P^*)\tau^a\phi(P^*))\frac{2g_Q}{f_\pi}(-i)\varepsilon^{\mu\nu\lambda\rho}p_\mu\varepsilon'_\nu q_\lambda\varepsilon_\rho, \end{aligned} \quad (2.28)$$

where ϕ^\dagger is the isospin wavefunction of the heavy meson anti-doublets, ε the polarization vector of P^* , and $p_\mu(q_\mu)$ the momentum of the heavy mesons (soft pion). We have used that $p_\mu \approx p'_\mu$ in the heavy-quark limit. Now PCAC implies

$$\begin{aligned} M(P^* \rightarrow P + \pi^a(q))_{PCAC} &= \frac{1}{f_\pi} \langle P | q^\mu A_\mu^a | P^* \rangle, \\ M(P^* \rightarrow P^* + \pi^a(q))_{PCAC} &= \frac{1}{f_\pi} \langle P^* | q^\mu A_\mu^a | P^* \rangle. \end{aligned} \quad (2.29)$$

Due to the heavy-quark spin decoupling, the Hilbert space of the four states, the 0^- - $|P\rangle$ state and the 3 spin states of the 1^- - $|P^*\rangle$, can be conveniently represented in a tensor product notation:⁵

$$|P, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle, \quad (2.30)$$

where the first $\pm\frac{1}{2}$ is the third component of the heavy-quark spin s_h , and the second $\pm\frac{1}{2}$ is that of the angular momentum of the light degrees of freedom s_m . In this notation,

the meson states are written as

$$\begin{aligned}
|P^*, +1\rangle &= \sqrt{2m_P} |P, +\frac{1}{2}, +\frac{1}{2}\rangle, \\
|P^*, 0\rangle &= \sqrt{2m_P} \frac{1}{\sqrt{2}} (|P, +\frac{1}{2}, -\frac{1}{2}\rangle + |P, -\frac{1}{2}, +\frac{1}{2}\rangle), \\
|P^*, -1\rangle &= \sqrt{2m_P} |P, -\frac{1}{2}, -\frac{1}{2}\rangle, \\
|P\rangle &= \sqrt{2m_P} \frac{1}{\sqrt{2}} (|P, +\frac{1}{2}, -\frac{1}{2}\rangle - |P, -\frac{1}{2}, +\frac{1}{2}\rangle).
\end{aligned} \tag{2.31}$$

The relative phase between the P^* and the P states is arbitrary, but that between the three P^* states is fixed by the angular momentum structure. Then, the $|P, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle$ state can be approximated by a product of the free heavy-quark state and a complicated brown-muck state:

$$|P, s_h, s_m\rangle \approx |h, s_h\rangle |\text{muck}, i_m, s_m\rangle, \tag{2.32}$$

with i_m the isospin of the brown muck associated with the light anti-quark. In the heavy-quark limit, this factorization becomes exact with the brown-muck state $|\text{muck}, i_m, s_m\rangle$ remaining the same for all states of Eq. (2.30) independently of the heavy quark spin s_h and flavor h .

Evaluating the matrix elements of the right-hand-side of Eq. (2.29) gives

$$\begin{aligned}
\langle P | q^\mu A_\mu^a | P^*, 0 \rangle &= m_P (\langle \text{muck}, i'_m, +\frac{1}{2} | q^\mu A_\mu | \text{muck}, i_m, +\frac{1}{2} \rangle \\
&\quad - \langle \text{muck}, i'_m, -\frac{1}{2} | q^\mu A_\mu | \text{muck}, i_m, -\frac{1}{2} \rangle), \\
\langle P^*, +1 | q^\mu A_\mu^a | P^*, +1 \rangle &= 2m_P \langle \text{muck}, i'_m, +\frac{1}{2} | q^\mu A_\mu | \text{muck}, i_m, +\frac{1}{2} \rangle.
\end{aligned}$$

With Wigner-Eckart theorem, the spin and isospin structure of A_μ given by Eq. (2.27) leads to

$$\langle \text{muck}, i'_m, s'_m | q^\mu A_\mu^a | \text{muck}, i_m, s_m \rangle = \alpha (\frac{1}{2} \phi'^\dagger \tau^a \phi) \langle s'_m | S^\mu | s_m \rangle q_\mu, \tag{2.33}$$

where S^μ is the spin operator of the brown muck and α is a constant independent of the heavy-quark mass. Thus

$$\begin{aligned}
\frac{1}{f_\pi} \langle P | q^\mu A_\mu^a | P^*, 0 \rangle &= \frac{1}{f_\pi} \alpha m_P (\frac{1}{2} \phi^\dagger(P) \tau^a \phi(P^*)) q_3, \\
\frac{1}{f_\pi} \langle P^*, +1 | q^\mu A_\mu^a | P^*, +1 \rangle &= \frac{1}{f_\pi} \alpha m_P (\frac{1}{2} \phi^\dagger(P) \tau^a \phi(P^*)) q_3.
\end{aligned} \tag{2.34}$$

Comparing Eq. (2.34) with Eq. (2.28) in the rest frame of P^* where the polarization vectors for the states $|P^*, +1\rangle$ and $|P^*, 0\rangle$, respectively, are $\varepsilon(+1) = \frac{1}{\sqrt{2}}(0, 1, +i, 0)$ and $\varepsilon(0) = (0, 0, 0, 1)$, we find

$$f_Q = \alpha m_P \quad \text{and} \quad g_Q = \frac{1}{2} \alpha, \tag{2.35}$$

that is,

$$g_Q = \frac{f_Q}{2m_P}.$$

Furthermore, Eq. (2.35) gives us a heavy-quark dependence of f_Q and g_Q :

$$f_Q = 2m_P g \quad \text{and} \quad g_Q = g, \quad (2.36)$$

with a universal constant g independent of the heavy-quark flavor. The coupling constant g can be evaluated using a nonrelativistic quark model (NRQM). In NRQM, the factorization (2.32) becomes much simpler:⁴⁴

$$|P, s_Q, s_\ell\rangle\rangle = |Q_{s_Q} \bar{q}_{s_\ell}\rangle,$$

with s_Q and s_ℓ the spins of the heavy quark and the light anti-quark. (Here, $|\rangle\rangle$ is to specify that it is a state in the nonrelativistic quark model.) For example, $|P\rangle$ and $|P^*, 0\rangle$ appearing in Eq. (2.34) can be written as

$$\begin{aligned} |P_{+\frac{1}{2}}^*, 0\rangle\rangle &= \sqrt{2m_P} \frac{1}{\sqrt{2}} [|Q_\uparrow \bar{d}_\downarrow\rangle + |Q_\downarrow \bar{d}_\uparrow\rangle], \\ |P_{-\frac{1}{2}}\rangle\rangle &= \sqrt{2m_P} \frac{1}{\sqrt{2}} [|Q_\uparrow \bar{u}_\downarrow\rangle - |Q_\downarrow \bar{u}_\uparrow\rangle], \end{aligned} \quad (2.37)$$

where the arrows represent the quark spin and the subscripts $\pm\frac{1}{2}$ the third component of the isospin. Since the axial current A_μ is defined in terms of the light quark doublet q as

$$A_i^a \equiv g_A^{ud} \frac{1}{2} \bar{q} \gamma_i \gamma_5 \tau^a q = \begin{cases} g_A^{ud} u^\dagger \sigma_i d & \text{if } a = 1 + i2, \\ \frac{1}{\sqrt{2}} g_A^{ud} (u^\dagger \sigma_i u - d^\dagger \sigma_i d) & \text{if } a = 3, \\ g_A^{ud} d^\dagger \sigma_i u & \text{if } a = 1 - i2, \end{cases}$$

with the axial vector coupling constant g_A^{ud} of the quark,⁴² the matrix element in the left-hand-side of Eq. (2.34) can be simply evaluated as

$$\begin{aligned} &\langle\langle P_{+\frac{1}{2}}^* | A_3^{1-i2} | P_{-\frac{1}{2}} \rangle\rangle \\ &= m_P g_A^{ud} \{ \langle \bar{u}_\downarrow | d^\dagger \sigma_3 u | \bar{d}_\downarrow \rangle - \langle \bar{u}_\uparrow | d^\dagger \sigma_3 u | \bar{d}_\uparrow \rangle \} \\ &= 2m_P g_A^{ud}, \end{aligned} \quad (2.38)$$

which implies that $g = -g_A^{ud}$ [$\alpha = -2$ in Eq. (2.33)]. A similar calculation leads to the matrix element of the axial vector current between the nucleon states:

$$\langle\langle N', S'_3 | A_i^a | N, S_3 \rangle\rangle = g_A^N \psi_{N', S'_3}^\dagger \tau^a \sigma_i \psi_{N, S_3}, \quad (2.39)$$

where ψ_{N, S_3} is the nucleon state ($N=p, n$) with spin $S_3 (= \pm\frac{1}{2})$. Therefore we get the familiar NRQM formula for the nucleon axial-vector coupling constant g_A^N

$$g_A^N = \frac{5}{3} g_A^{ud}. \quad (2.40)$$

The experimental value $g_A^N = 1.25$ determines the value for g_A^{ud}

$$g = -g_A^{ud} = -0.75. \quad (2.41)$$

Higher-order corrections in $1/N_c$ to g_A^{ud} are discussed in Ref. 46.

In the case of $Q(\text{or } h)=c$, the matrix elements (2.28) determine the decay width

$$\Gamma(D^{*+} \rightarrow D^0 \pi^+) = \frac{1}{12\pi} \frac{g^2}{f_\pi^6} |\vec{p}_\pi|^3. \quad (2.42)$$

The width for $D^{*+} \rightarrow D^+ \pi^0$ is reduced by a factor of two due to isospin symmetry. The experimental upper limit⁴⁷ of 131 KeV on the D^* width when combined with the $D^{*+} \rightarrow D^+ \pi^0$ and $D^{*+} \rightarrow D^0 \pi^+$ branching ratios⁴⁸ implies that $|g|^2 \lesssim 0.5$. The nonrelativistic quark model prediction on g is slightly bigger than the experimental upper limit but is consistent with this value.

3. Skyrme Model and Hidden Symmetry: “Bottom-Up” Approach

Thus far, we have discussed the heavy-quark chiral Lagrangian in the heavy-quark symmetry limit with $m_Q \rightarrow \infty$. Restricting ourselves to two massless flavors and one heavy flavor, the leading order Lagrangian is constructed by taking into account both $SU(2) \times SU(2)$ chiral symmetry and the heavy-quark symmetry. For the actual hadrons with finite heavy-quark mass as relevant to the c and b quarks, we need to include the deviation from the symmetry limit which starts with order $1/m_Q$ corrections. In this section, we approach the heavy-quark regime from “below.”

The Skyrme model⁴⁹ describes a baryon as a soliton solution of a nonlinear chiral Lagrangian of weakly interacting Goldstone bosons. Since an exact bosonization of fermionic theories has not yet been found in (3+1) dimensions, such a bosonic Lagrangian describing QCD does not exist except in the limit that the number of colors is infinite. Nonetheless the soliton approach based on approximate effective Lagrangians has enjoyed a great success ranging from the static properties of the baryons to the inter-nucleon interactions. (For review, see Refs. 50,51,52,53,54,55,56.) We wish to show below that this model, with a minimal complication, can provide an amazingly simple way of constructing such a heavy meson effective Lagrangian by starting from the $SU(3)_L \times SU(3)_R$ ^{#2} chiral limit and then climbing up, *à la* CK approach, to the massive system with the symmetry breaking to $SU(2) \times U(1)$.

Suppose that we start with three massless quarks, assuming the spontaneous breaking of chiral $SU(3)_L \times SU(3)_R$ down to the $SU(3)_V$ vector symmetry. We write the chiral field as

$$U = \exp\left(\frac{i}{f_\pi} \sum_{a=1}^8 \lambda_a \pi_a\right) = e^{iM/f_\pi}, \quad (3.1)$$

where $\lambda_a (a=1,2,\dots,8)$ is the Gell-Mann matrices for flavor $SU(3)$ and M is the $SU(3)$ -valued meson field:

$$M = \begin{bmatrix} \pi^0 + \frac{1}{\sqrt{3}}\Phi & \sqrt{2}\pi^+ & P^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\Phi & P^0 \\ P^- & \bar{P}^0 & -\frac{2}{\sqrt{3}}\Phi \end{bmatrix}. \quad (3.1a)$$

^{#2}Since we are interested in the hadron system which contains one heavy flavor, it is not necessarily the conventional $SU(3)$ with u , d and s flavors. It could be generalized to any $SU(3)$ subgroup of the full $SU(N_f)$ ($N_f \geq 3$) associated with the u , d and h flavor of our interest.

Here, P^+ , P^0 , P^- , \bar{P}^0 and Φ denote the mesons with the quantum numbers of $\bar{h}\gamma_5 u$, $\bar{h}\gamma_5 d$, $\bar{u}\gamma_5 h$ and $\bar{d}\gamma_5 h$ and $\bar{u}\gamma_5 u + \bar{d}\gamma_5 d - 2\bar{h}\gamma_5 h$, respectively. For example, if $h=s$, they correspond to K^+ , K^0 , K^- , \bar{K}^0 and η_8 . The Lagrangian for interactions among the Goldstone bosons is given by generalizing Eq. (2.8) to three flavors, where the Wess-Zumino term figures crucially:

$$\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \cdots + \mathcal{L}_{WZ}. \quad (3.2)$$

What we are interested in is the situation where the symmetry $SU(3)_L \times SU(3)_R$ is *explicitly* broken to $SU(2)_L \times SU(2)_R \times U(1)$ by an h -quark mass^{#3}, thereby making the P -meson massive and its decay constant f_P different from that of the pion. These two effects of symmetry breaking can be effectively incorporated into the Lagrangian^{26,57} by a term of the form

$$\begin{aligned} \mathcal{L}_{SB} = & \frac{1}{6} f_P^2 m_P^2 \text{Tr}[(1 - \sqrt{3}\lambda_8)(U + U^\dagger - 2)] \\ & + \frac{1}{12}(f_P^2 - f_\pi^2) \text{Tr}[(1 - \sqrt{3}\lambda_8)(U \partial_\mu U^\dagger \partial^\mu U + U^\dagger \partial_\mu U \partial^\mu U^\dagger)]. \end{aligned} \quad (3.3)$$

The appropriate ansatz for the chiral field is the Callan-Klebanov (CK)-type which we shall take in the form

$$U = N_\pi U_P N_\pi, \quad (3.4)$$

where

$$N_\pi = \exp\left(\frac{i}{2f_\pi} \sum_{a=1}^3 \lambda_a \pi_a\right) = \begin{bmatrix} \xi & 0 \\ 0 & 1 \end{bmatrix}, \quad (3.4a)$$

$$U_P = \exp\left(\frac{i}{f_\pi} \sum_{a=4}^7 \lambda_a \pi_a\right) = \exp\left(\frac{i\sqrt{2}}{f_\pi} \begin{bmatrix} \mathbf{0} & P^\dagger \\ P & 0 \end{bmatrix}\right), \quad (3.4b)$$

with $SU(2)$ matrix ξ defined by Eq. (2.13), the P -meson anti-doublets $P = (P^-, \bar{P}^0)$, and P -meson doublets $P^\dagger = (P^+, \bar{P}^0)^T$. One can easily see that ξ and P transform exactly in the same way as Eqs. (2.14) and (2.17) under the embedded $SU(2)_L \times SU(2)_R$ rigid chiral transformation

$$U \rightarrow U' = \begin{bmatrix} L & 0 \\ 0 & 1 \end{bmatrix} U \begin{bmatrix} R^\dagger & 0 \\ 0 & 1 \end{bmatrix}$$

with $L \in SU(2)_L$ and $R \in SU(2)_R$.

Substituting the CK ansatz (3.4) into the Lagrangian (3.2) with the symmetry breaking term (3.3) and expanding up to second order in the P -meson field, we obtain

$$\mathcal{L} = \mathcal{L}_M + D_\mu P D_\mu P^\dagger - M_P^2 P P^\dagger - P A_\mu^\dagger A^\mu P^\dagger - \frac{iN_c}{4f_P^2} B_\mu (D^\mu P P^\dagger - P D^\mu P^\dagger), \quad (3.5)$$

^{#3}For simplicity, we turn off the light quark masses.

where we have rescaled the P -meson fields as P/χ with $\chi = f_P/f_\pi$. The covariant derivative $D_\mu P^\dagger$ is $(\partial_\mu + V_\mu)P^\dagger$, the vector field V_μ and the axial-vector field A_μ are the same as in the Lagrangian (2.19), and B_μ is the topological current

$$B^\mu = \frac{1}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} \text{Tr}(U^\dagger \partial_\nu U U^\dagger \partial_\lambda U U^\dagger \partial_\rho U), \quad (3.6)$$

which is the baryon number current in the Skyrme model. One can see that as far as the P -fields are concerned, to the lowest order in derivative on the Goldstone boson fields, Eq. (3.5) is the same as the Lagrangian Eq. (2.19). Furthermore as argued by Nowak *et al.*,⁵⁸ one expects that as the h quark mass increases above the chiral scale Λ_χ , the Wess-Zumino term would vanish, thereby turning off the last term of (3.5). Thus the two Lagrangians are indeed equivalent as far as the pseudoscalars are concerned.

As mentioned above, going to heavy-quark systems requires the vector degrees of freedom which become degenerate with the pseudoscalars in the infinite quark mass limit. From a chiral Lagrangian point of view, the vector mesons can be viewed as “matter fields” and there are several ways of introducing matter fields in general. If chiral symmetry is correctly implemented, they are all equivalent in the sense that the S -matrix is identical. When anomalies are involved, the situation is a bit delicate but by now there is no conceptual difficulty.^{53,59,60,61,62,63} Here we follow the hidden gauge symmetry (HGS) approach^{60,61} which in our opinion offers psychologically the most powerful one.

The chiral field U in the Lagrangian (3.2) transforms

$$U \rightarrow U' = LUR^\dagger \quad (L \in [SU(3)_L]_{\text{global}} \quad R \in [SU(3)_R]_{\text{global}}).$$

The hidden gauge symmetry $SU(3)_V$ of the Lagrangian (3.2) can be made apparent by rewriting U in terms of two $SU(3)$ matrices $\xi_{L,R}(x)$ as

$$U(x) = \xi_L^\dagger(x) \cdot \xi_R(x). \quad (3.7)$$

The Lagrangian is invariant under $[SU(3)_L \times SU(3)_R]_{\text{global}} \times [SU(3)_V]_{\text{local}}$ transformations:

$$\begin{aligned} \xi_L(x) &\rightarrow \xi'_L(x) = h(x)\xi_L(x)L^\dagger, \\ \xi_R(x) &\rightarrow \xi'_R(x) = h(x)\xi_R(x)R^\dagger, \end{aligned} \quad (3.8)$$

with $h(x) \in [SU(3)_V]_{\text{local}}$. The gauge connection associated with the $SU(3)_V$ local symmetry can be written as

$$U_\mu = \frac{1}{2} \begin{bmatrix} \omega_\mu + \rho_\mu \sqrt{2} P_\mu^{*\dagger} \\ \sqrt{2} P_\mu^* & \Phi_\mu^* \end{bmatrix}. \quad (3.9)$$

It transforms as

$$U_\mu \rightarrow U'_\mu = h(x)U_\mu(x)h^\dagger(x) - \frac{i}{g}h(x)\partial_\mu h^\dagger(x). \quad (3.9a)$$

The covariant derivative relevant to the HGS is then

$$\mathcal{D}_\mu \xi_{L,R} \equiv (\partial_\mu + ig_* U_\mu) \xi_{L,R}, \quad (3.9b)$$

with a gauge coupling constant g_* to be specified later. Up to second order in this covariant derivative, we can construct two independent terms consistent with the $[SU(3)_L \times SU(3)_R]_{\text{global}} \times [SU(3)_V]_{\text{local}}$ symmetry and parity:

$$\begin{aligned} \mathcal{L}_V &\equiv -\frac{1}{4}f_\pi^2 \text{Tr}[\mathcal{D}_\mu \xi_L \xi_L^\dagger + \mathcal{D}_\mu \xi_R \xi_R^\dagger]^2, \\ \mathcal{L}_A &\equiv -\frac{1}{4}f_\pi^2 \text{Tr}[\mathcal{D}_\mu \xi_L \xi_L^\dagger - \mathcal{D}_\mu \xi_R \xi_R^\dagger]^2. \end{aligned} \quad (3.10)$$

In working at tree order, we might as well choose the unitary gauge

$$\xi_L^\dagger = \xi_R \equiv \xi(x), \quad \xi^2(x) = U(x). \quad (3.11)$$

Then, we have

$$\begin{aligned} \mathcal{L}_A &= -\frac{1}{4}f_\pi^2 \text{Tr}[\partial_\mu \xi \xi^\dagger - \partial_\mu \xi^\dagger \xi]^2 = \frac{1}{4}f_\pi^2 \text{Tr}(\partial_\mu U^\dagger \partial^\mu U), \\ \mathcal{L}_V &= f_\pi^2 \text{Tr}[g_* U_\mu - \frac{1}{2}i(\partial_\mu \xi \xi^\dagger + \partial_\mu \xi^\dagger \xi)]^2. \end{aligned}$$

Thus \mathcal{L}_A reduces to the original Lagrangian while \mathcal{L}_V vanishes identically with U_μ satisfying the equation of motion

$$U_\mu = \frac{i}{2g_*}(\partial_\mu \xi \xi^\dagger + \partial_\mu \xi^\dagger \xi). \quad (3.12)$$

Clearly nothing is gained by “gauging” the hidden local symmetry. The gauge field is just an auxiliary field. However the dynamics changes dramatically if the gauge field becomes a propagating field by acquiring a kinetic energy term. The kinetic term is higher order in derivative and chirally invariant. Therefore a systematic chiral expansion would naturally allow such a term. Were we to attempt to derive a HGS Lagrangian from QCD by functional integration of the gluon and quark fields in the presence of auxiliary vector fields, then one would naturally encounter such a term from the fermion determinant. (This is obvious by “bosonizing” the Nambu–Jona-Lasinio (NJL) to an effective bosonic Lagrangian with vector mesons implemented.) The resulting Lagrangian is

$$\mathcal{L}_0 = \mathcal{L}_A + a\mathcal{L}_V - \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}), \quad (3.13)$$

with the field strength tensor of the vector mesons

$$F_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu + ig_*[U_\mu, U_\nu].$$

The vector meson mass M_V and the $\rho\pi\pi$ coupling constant can be read off from the Lagrangian,

$$\begin{aligned} m_V^2 &= ag_*^2 f_\pi^2, \\ g_{\rho\pi\pi} &= \frac{1}{2}ag_*. \end{aligned} \quad (3.14)$$

With the KSFR relation⁶⁴

$$m_\rho^2 = 2g_*^2 f_\pi^2, \quad (3.15)$$

and the universality of the vector-meson coupling

$$g_{\rho\pi\pi} = g_*, \quad (3.16)$$

we can fix the arbitrary parameter a to 2.

[We should parenthetically mention an interesting situation that arises in the limit $M_V^2 \rightarrow \infty$ with g_* held fixed.⁶⁵ In this limit, we have the constraint

$$U_\mu = \frac{i}{2g_*}(\partial_\mu \xi_L \xi_L^\dagger + \partial_\mu \xi_R \xi_R^\dagger).$$

When this constraint is substituted into the field strength tensor $F_{\mu\nu}$, we get

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{4ig_*}[(\partial_\mu \xi_L \xi_L^\dagger - \partial_\mu \xi_R \xi_R^\dagger), (\partial_\nu \xi_L \xi_L^\dagger - \partial_\nu \xi_R \xi_R^\dagger)] \\ &= \frac{i}{4g_*} \xi_L [\partial_\mu U U^\dagger, \partial_\nu U U^\dagger] \xi_L^\dagger. \end{aligned}$$

Thus, the kinetic term for the vector meson becomes the Skyrme term in the Skyrme Lagrangian

$$\mathcal{L}_V^{\text{kin}} \xrightarrow{m_V^2 \rightarrow \infty} \mathcal{L}_{Sk} = +\frac{1}{32g_*^2} \text{Tr}[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2, \quad (3.17)$$

with $e=g_*$ the Skyrme parameter.⁴⁹

There are two remarkable features associated with this term. First it renders the soliton stable against the Hobard-Derrick collapse whereas vector mesons for *any* finite vector-meson mass cannot. This means that the stabilization occurs only in the limit that the mass is infinite. The second observation is that when the energy density of a dense hadronic matter is computed for a multi-skyrmion system at asymptotic density, it is found to be identical to that of the free quark gas. This implies that the quartic Skyrme term possesses short-distance physics described by asymptotically free quarks. This may be the reason why the Skyrme quartic term can stabilize the soliton while finite mass vectors cannot.]

The effective action should satisfy the same anomalous Ward identities as does the underlying fundamental theory QCD.⁴¹ In the presence of vector mesons, $A_{L,R}^\mu$ associated with the external gauge transformation $U(x) \rightarrow e^{i\varepsilon_L} U(x) e^{-i\varepsilon_R}$ and U_μ with the hidden gauge transformation discussed so far, the Wess-Zumino anomaly equation reads

$$\delta\Gamma(\xi_L, \xi_R, A_H, A_L, A_R) = -\frac{N_c}{24\pi^2} \int_{M_4} \text{Tr}[\varepsilon_L \{(dA_L)^2 - \frac{1}{2} idA_L^3\} - \{L \leftrightarrow R\}], \quad (3.18)$$

where the gauge transformation δ is

$$\delta = \delta_L(\varepsilon_L) + \delta_R(\varepsilon_R) + \delta_H(h). \quad (3.18a)$$

Here the subscript H stands for hidden gauge symmetry. For convenience, we use the differential one-form notations:

$$\begin{aligned} A_H &\equiv U_\mu dx^\mu, \\ A_{L,R} &\equiv A_{L,R}^\mu dx_\mu. \end{aligned} \quad (3.18b)$$

The general solution to Eq. (3.18) is given by a special solution plus general solutions of the homogeneous equation $\delta\Gamma = 0$. The former is the Wess-Zumino action (2.10) (see Ref. 53,61 for details) and the latter, the anomaly free terms, can be made of gauge-covariant building blocks. There are six independent forms⁶⁶ that conserve parity but violate intrinsic parity^{#4}:

$$\begin{aligned} \mathcal{L}_1 &= \text{Tr}[\hat{L}^3 \hat{R} - \hat{R}^3 \hat{L}], & \mathcal{L}_2 &= \text{Tr}[\hat{L} \hat{R} \hat{L} \hat{R}], \\ \mathcal{L}_3 &= ig \text{Tr}[F_H(\hat{L}^2 - \hat{R}^2)], & \mathcal{L}_4 &= ig \text{Tr}[F_H(\hat{L} \hat{R} - \hat{R} \hat{L})], \\ \mathcal{L}_5 &= i \text{Tr}[\hat{F}_L \hat{R}^2 - \hat{F}_R \hat{L}^2], & \mathcal{L}_6 &= i \text{Tr}[\hat{F}_L \hat{L} \hat{R} - \hat{F}_R \hat{R} \hat{L}], \end{aligned} \quad (3.19)$$

with the differential one-forms defined as

$$\begin{aligned} \hat{L}, \hat{R} &\equiv \mathcal{D}\xi_{L,R} \xi_{L,R}^\dagger = d\xi_{L,R} \xi_{L,R}^\dagger - ig A_H + i \xi_{L,R} A_{L,R} \xi_{L,R}^\dagger, \\ F_H &\equiv dA_H + ig A_H^2, \\ \hat{F}_{L,R} &= \xi_{L,R} (dA_{L,R} - i A_{L,R}^2) \xi_{L,R}^\dagger. \end{aligned}$$

Thus, for the intrinsic parity violation processes, we have

$$\Gamma = \Gamma_{WZ}[\xi_L^\dagger \xi_R, A_L, A_R] + \sum_{i=1}^6 c_i \int_{M_4} \mathcal{L}_i, \quad (3.20)$$

with 6 arbitrary constants c_i , which are determined by experimental data. Vector meson dominance (VMD) in the process like $\pi^0 \rightarrow 2\gamma$ and $\gamma \rightarrow 3\pi$ is very useful in determining the constants. In Table 3.1 a few sets of constants used in the literature are listed.

The $[SU(3)_L \times SU(3)]_{\text{global}}$ breaking term can be incorporated without affecting the hidden symmetry by writing $\mathcal{L}_{A,V}$ as⁶⁸

$$\begin{aligned} \mathcal{L}_A &= -\frac{1}{4} f_\pi^2 \text{Tr}\{(\mathcal{D}_\mu \xi_L \xi_L^\dagger + \mathcal{D}_\mu \xi_L \varepsilon_A \xi_R^\dagger) + (\mathcal{D}_\mu \xi_R \xi_R^\dagger + \mathcal{D}_\mu \xi_R \varepsilon_A \xi_L^\dagger)\}^2, \\ \mathcal{L}_V &= -\frac{1}{4} f_\pi^2 \text{Tr}\{(\mathcal{D}_\mu \xi_L \xi_L^\dagger + \mathcal{D}_\mu \xi_L \varepsilon_V \xi_R^\dagger) - (\mathcal{D}_\mu \xi_R \xi_R^\dagger + \mathcal{D}_\mu \xi_R \varepsilon_V \xi_L^\dagger)\}^2, \end{aligned} \quad (3.21)$$

with the covariant derivative given by Eq. (3.9b). The symmetry breaking matrices $\varepsilon_{A,V}$ are defined by

$$\varepsilon_{A,V} = c_{A,V} \frac{1}{3} (1 - \sqrt{3} \lambda_8) \quad (3.22)$$

^{#4}The intrinsic parity of a particle is defined to be even if its parity equals $(-1)^{\text{spin}}$. Considering C -parity, \mathcal{L}_3 and \mathcal{L}_5 may be discarded and \mathcal{L}_6 should read $\frac{i}{2} \text{Tr}\{\hat{F}_L[\hat{L}, \hat{R}] - \hat{F}_R[\hat{R}, \hat{L}]\}$. See Ref. 53 for details.

Table 3.1 : Constants c_i in Eq. (3.20). ($C=-iN_c/240\pi^2$ and $\alpha - \beta=1$)

	c_1	c_2	c_3	c_4	c_5	c_6
PVMD ^{*66}	$15C\alpha$	$15C\beta$	0	$-15C$	0	$-15C$
CVMD ^{†66}	$5C\alpha$	$5C\beta$	0	$-15C$	0	$-15C$
Minimal Model ^{‡67}	$10C$	$-10C$	0	0	0	0

* Partial VMD with no contact term for the $\omega \rightarrow 3\pi$ decay.

† Complete VMD.

‡ VMD in the isoscalar channel.

with $c_{A,V}$ being constants. These additional terms lead to a renormalization of the Goldstone boson field M of Eq. (3.1a) through

$$\sqrt{1 + \varepsilon_A} M \sqrt{1 + \varepsilon_A} \rightarrow M,$$

and thus modify the vector meson masses as

$$m_\rho^2 = m_\omega^2 = ag^2 f_\pi^2 = \frac{m_{P^*}^2}{1 + c_V} = \frac{m_{\Phi^*}^2}{(1 + c_V)^2}, \quad (3.23)$$

and the meson decay constants as

$$f_P = f_\pi \sqrt{1 + c_A}. \quad (3.24)$$

In the case of $h=s$ ($P^*=K^*$ and $\Phi^*=\phi$), the relation derived from Eq. (3.23)

$$\frac{m_\phi}{m_{K^*}} = \frac{m_{K^*}}{m_\rho} = \sqrt{1 + c_V} \sim 1.15$$

holds within 2% and, furthermore, $c_A=c_V$ results in

$$f_K/f_\pi = m_{K^*}/m_\rho \sim 1.15,$$

in good agreement with experimental data. However, for the heavier flavor, neither $c_A = c_V$ nor $m_{\Phi^*}/m_{P^*} = m_{P^*}/m_\rho$ holds well. (See Table 3.2.)

One may introduce the symmetry breaking in a different way from that of Eq. (3.21). Note that, while its role is the same in modifying the meson decay constant as the second term of Eq. (3.3), the additional terms in \mathcal{L}_A cannot be reduced to the same form. We can rewrite the latter in terms of ξ_L and ξ_R by substituting $U = \xi_L^\dagger \xi_R$:

$$\begin{aligned} & -\frac{1}{12}(f_K^2 - f_\pi^2) \text{Tr} \left\{ (1 - \sqrt{3}\lambda_8)(U \partial_\mu U^\dagger \partial^\mu U + U^\dagger \partial_\mu U \partial^\mu U^\dagger) \right\} \\ & = -\frac{1}{4}f_\pi^2 \text{Tr} \left\{ (\partial_\mu \xi_L \xi_L^\dagger - \partial_\mu \xi_R \xi_R^\dagger)(\xi_R \varepsilon_A \xi_L^\dagger + \xi_L \varepsilon_A \xi_R^\dagger)(\partial^\mu \xi_L \xi_L^\dagger - \partial^\mu \xi_R \xi_R^\dagger) \right\}. \end{aligned}$$

Table 3.2 : Symmetry breakings in the vector meson masses (in MeV)
and in the meson decay constants.

h	m_{P^*}	$\frac{m_{P^*}}{m_\rho}$	c_V	$m_{\Phi^*}^{a)}$	$m_{\Phi^*}^{b)}$	$m_{\Phi^*}^{c)}$	f_P/f_π	c_A
s	892	1.161	0.349	1020	1036	1001	1.22	0.488
c	2010	2.617	5.850	3097	5260	2736	1.80	2.240
b	5325	6.934	47.07	9460	3.7×10^3	7491	-	-

^{a)} experimental data, ^{b)} Eq. (3.23) and ^{c)} Eq. (3.26).

It suggests introducing the following symmetry breaking into the Lagrangian^{23,69}

$$\begin{aligned}\mathcal{L}_A &= -\frac{1}{4}f_\pi^2 \text{Tr} \left\{ (\mathcal{D}_\mu \xi_L \xi_L^\dagger - \mathcal{D}_\mu \xi_R \xi_R^\dagger) (1 + \xi_R \varepsilon_A \xi_L^\dagger + \xi_L \varepsilon_A \xi_R^\dagger) (\mathcal{D}^\mu \xi_L \xi_L^\dagger - \mathcal{D}^\mu \xi_R \xi_R^\dagger) \right\}, \\ \mathcal{L}_V &= -\frac{1}{4}f_\pi^2 \text{Tr} \left\{ (\mathcal{D}_\mu \xi_L \xi_L^\dagger + \mathcal{D}_\mu \xi_R \xi_R^\dagger) (1 + \xi_R \varepsilon_V \xi_L^\dagger + \xi_L \varepsilon_V \xi_R^\dagger) (\mathcal{D}^\mu \xi_L \xi_L^\dagger + \mathcal{D}^\mu \xi_R \xi_R^\dagger) \right\}.\end{aligned}\tag{3.25}$$

Compared with Eq. (3.21), Eq. (3.25) can be understood as a linearized version of it.^{#5} Although not exact, it corresponds to eliminating the terms with $\varepsilon_{A,V}^2$ in the expansion of Eqs. (3.21). Note that the disastrous Υ mass in Eq. (3.23) is mainly due to the c_V^2 in the denominator. As a consequence, \mathcal{L}_V leads to the vector meson masses

$$m_{\rho,\omega}^2 = ag^2 f_\pi^2 = \frac{m_{P^*}^2}{1 + c_V} = \frac{m_{\Phi^*}^2}{1 + 2c_V},\tag{3.26}$$

which satisfy Gell-Mann–Okubo mass formula

$$m_{\Phi^*}^2 - m_{P^*}^2 = m_{P^*}^2 - m_{\rho,\omega}^2,$$

and yield a lot more reasonable masses for the vector mesons ϕ , J/ψ and Υ than Eq. (3.23). (See Table 3.2.)

Finally, we substitute the CK ansatz (3.4), $U = N_\pi U_P N_\pi$ (that is, $\xi_L^\dagger = N_\pi \sqrt{U_P}$ and $\xi_R = \sqrt{U_P} N_\pi$ ^{#6}), into the Lagrangian constructed so far in the form of $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{an} + \mathcal{L}_{SB}$ with \mathcal{L}_0 , \mathcal{L}_{an} and \mathcal{L}_{SB} given by Eq. (3.13), Eq. (3.20) and Eq. (3.25), respectively. The resulting Lagrangian reads

$$\mathcal{L} = \mathcal{L}_{SU(2)} + \mathcal{L}_{PP^*}^n + \mathcal{L}_{PP^*}^{an},\tag{3.27}$$

^{#5}It is also similar to (but not equal to) the α -type symmetry breaking term discussed in Ref. 70. Eq. (3.25) can be regarded as a generalization of their α -type term.

^{#6}This does not exactly correspond to the standard unitary gauge, $\xi_L^\dagger = \xi_R$, unless $N_K=1$. Different ansätze correspond roughly to different gauge choices and we may think of this as one particular gauge. However, it is plausible that the results do not depend on these gauge choices.¹⁸

with

$$\begin{aligned}
\mathcal{L}_{SU(2)} = & \frac{1}{4}f_\pi^2 \text{tr}(\partial_\mu U_\pi^\dagger \partial^\mu U_\pi) - \frac{1}{2} \text{tr}(q_{\mu\nu} q^{\mu\nu}) \\
& + m_\rho^2 \text{tr}(q_\mu + \frac{i}{g_*} V_\mu)^2 + 6\pi^2(c_1 - c_2) i g_* \omega_\mu B^\mu \\
& - \frac{1}{2} g_*^3 (c_1 + c_2) \{ -\varepsilon^{\mu\nu\alpha\beta} \omega_\mu \text{tr}(A_\nu \bar{\rho}_\alpha \bar{\rho}_\beta) + \varepsilon^{\mu\nu\alpha\beta} \text{tr}(A_\mu \bar{\rho}_\nu \bar{\rho}_\alpha \bar{\rho}_\beta) \} \\
& + i g_* c_4 \left\{ -\varepsilon^{\mu\nu\alpha\beta} \omega_\mu \text{tr}(q_{\nu\alpha} A_\beta) + \varepsilon^{\mu\nu\alpha\beta} \text{tr}\{q_{\mu\nu}(A_\alpha \bar{\rho}_\beta - \bar{\rho}_\alpha A_\beta)\} \right\},
\end{aligned} \tag{3.27a}$$

$$\begin{aligned}
\mathcal{L}_{PP^*}^n = & D_\mu P D^\mu P^\dagger - m_P^2 K K^\dagger - P A_\mu A^\mu P^\dagger \\
& + \frac{f_\pi^2}{f_P^2} \left[P(V_\mu - i g_* q_\mu) D^\mu P^\dagger - \mathcal{D}_\mu P (V_\mu - i g_* q_\mu) P^\dagger \right] \\
& - \frac{1}{2} \left[P^{*\mu\nu} P_{\mu\nu}^{*\dagger} - 2g_*^2 P_\mu^* q^{\mu\nu} P_\nu^{*\dagger} \right] \\
& + m_{P^*}^2 \left[P^{*\mu} - \frac{\sqrt{2}}{m_{P^*}} P A^\mu \right] \left[P_\mu^{*\dagger} - \frac{\sqrt{2}}{m_{P^*}} A_\mu P^\dagger \right],
\end{aligned} \tag{3.27b}$$

$$\begin{aligned}
\mathcal{L}_{PP^*}^{an} = & -\frac{iN_c}{8f_P^2} B_\mu \{ D^\mu P P^\dagger - P D_\mu P^\dagger \} \\
& + (c_1 - c_2) \left\{ -\frac{3\pi^2}{f_P^2} B_\mu (D^\mu P P^\dagger - P D^\mu P^\dagger) + \dots \right\} \\
& - (c_1 + c_2) \left\{ -\frac{2}{f_P^2} g_*^3 \varepsilon^{\mu\nu\alpha\beta} \omega_\mu P A_\nu \bar{\rho}_\alpha \bar{\rho}_\beta P^\dagger + \dots \right\} \\
& + i c_4 \left\{ \frac{1}{2} g_*^2 \varepsilon^{\mu\nu\lambda\rho} (P_{\mu\nu}^* A_\lambda P_\rho^{*\dagger} - P_\mu^* A_\nu P_{\lambda\rho}^{*\dagger}) \right. \\
& \quad \left. - i g_* \frac{1}{f_P} \varepsilon^{\mu\nu\lambda\rho} (D_\mu P q_{\nu\lambda} P_\rho^{*\dagger} + P_\mu^* q_{\nu\lambda} D_\rho P^\dagger) + \dots \right\},
\end{aligned} \tag{3.27c}$$

where

$$\begin{aligned}
D_\mu P^\dagger &= (\partial_\mu + V_\mu) P^\dagger, \\
q_\mu &= \frac{1}{2}(\omega_\mu + \rho_\mu) = \frac{1}{2} \begin{bmatrix} \omega_\mu + \rho_\mu^0 & \sqrt{2}\rho_\mu^+ \\ \sqrt{2}\rho_\mu^- & \omega_\mu - \rho_\mu^0 \end{bmatrix}, \\
q_{\mu\nu} &= \frac{1}{2}(\omega_{\mu\nu} + \rho_{\mu\nu}) = \partial_\mu q_\nu - \partial_\nu q_\mu + i g_* [q_\mu, q_\nu], \\
P_{\mu\nu}^{*\dagger} &= (\partial_\mu + i g_* q_\mu) P_\nu^{*\dagger} - (\partial_\nu + i g_* q_\nu) P_\mu^{*\dagger}, \\
\bar{\rho}_\mu &= \rho_\mu + i 2 V_\mu / g_*, \\
B^\mu &= \frac{1}{24\pi^2} \varepsilon^{\mu\nu\lambda\rho} \text{tr}(U_\pi^\dagger \partial_\nu U_\pi U_\pi^\dagger \partial_\lambda U_\pi U_\pi^\dagger \partial_\rho U_\pi).
\end{aligned} \tag{3.27d}$$

One may check that Eq. (3.27) contains all the terms of Eq. (2.19). Explicitly, we have

$$\begin{aligned}
\mathcal{L} = & \frac{f_\pi^2}{4} \text{tr}(\partial_\mu U_\pi^\dagger \partial^\mu U_\pi) + \frac{1}{32g_*^2} \text{tr}[U_\pi^\dagger \partial_\mu U_\pi, U_\pi^\dagger \partial_\nu U_\pi]^2 \\
& + D_\mu P D_\mu P^\dagger - m_P^2 P P^\dagger - \frac{1}{2} P^{*\mu\nu} P_{\mu\nu}^{*\dagger} + m_{P^*}^2 P^{*\mu} P_\mu^{*\dagger} \\
& - \sqrt{2} m_{P^*} (P A^\mu P_\mu^{*\dagger} + P_\mu^* A^\mu P^\dagger) + \frac{i}{2} c_4 g_*^2 \varepsilon^{\mu\nu\lambda\rho} (P_{\mu\nu}^* A_\lambda P_\rho^{*\dagger} + P_\lambda^* A_\rho P_{\mu\nu}^{*\dagger}) + \dots,
\end{aligned} \tag{3.28}$$

where we have replaced the light vector meson fields ρ_μ and ω_μ by $i2V_\mu/g_*$ and $(c_1 - c_2)i6\pi^2 B_\mu/g_* f_\pi^2$ respectively and kept the leading order terms in m_P and m_{P^*} with a single derivative on the pion fields. Comparing Eq. (3.28) with Eq. (2.19), we get two relations:

$$f_Q = -\sqrt{2}m_{P^*}, \quad \text{and} \quad g_Q = ic_4 g_*^2.$$

The first relation implies (see Eq. (2.36)) that the g_Q value is

$$g_Q = -\frac{1}{\sqrt{2}} \simeq -0.71, \quad (3.29)$$

which is quite close to $g_Q = -0.75$ evaluated with the NRQM in Sec. 2. If one assumes that the VMD works in the heavy meson sector for which c_4 is fixed to $iN_c/16\pi^2$, we could obtain g_* in the heavy quark limit: *viz.*

$$g_* = \sqrt{\frac{16\pi^2}{\sqrt{2}N_c}} \simeq 6 \quad (\text{with } N_c=3). \quad (3.30)$$

It is intriguing that this is so close to $g_* = g_{\rho\pi\pi} (=6.11)$ found in the light meson sector.

4. Effective Field Theory for Heavy Mesons

4.1. Heavy-Quark Effective Theory

At this point, we digress a bit to discuss in a simple picture^{5,71} how the heavy quark symmetries arise in the hadronic processes involving a heavy quark. Consider two hadrons A and B , each of which is made of a single heavy quark of mass m_Q^A and m_Q^B , respectively, and the light degrees of freedom. If the heavy quark masses are much larger than the scales of QCD interactions, $m_Q^A, m_Q^B \gg \Lambda_{QCD}$, then in the rest frame of the heavy quark, how QCD distributes the light degrees of freedom around the static heavy quark is independent of the heavy flavor; *i.e.*, as far as the color charge is concerned, the light degrees of freedom do not know the difference. This feature induces a new $SU(N_h)$ flavor symmetry for N_h flavors of heavy quark. By boosting one can extend the heavy flavor symmetry to any heavy quarks of the same *velocity*. This is the reason why the velocity (not the momentum) has a physical significance in the heavy quark theory. Furthermore, the heavy quark spin decouples as $1/m_Q$ when the heavy quark mass becomes infinitely large.

The heavy system that we are considering here is a QCD bound state of a heavy quark and light quarks (and/or light anti-quarks), where the heavy quark carries most, but not all, of the momentum. Consider, for example, a heavy meson moving with a 4-velocity v^μ ($v_\mu v^\mu = 1$, $v^0 > 0$). Due to its huge momentum, its evolution is classical and the 4-momentum of the bound state has the form of

$$P_{bs}^\mu = m_P v^\mu, \quad (4.1)$$

where the heavy meson mass m_P is essentially the same as the heavy quark mass m_Q

$$m_P \approx m_Q,$$

and the difference is expected to be independent of m_Q . Let the small momentum of the light degrees of freedom and the residual momentum of the heavy quark be q^μ and k^μ , respectively. Then, we can write the momentum of the heavy quark as

$$p_{hq}^\mu = P_{bs}^\mu - q^\mu = m_Q v^\mu + k^\mu. \quad (4.2)$$

The 4-velocity of the heavy quark can be defined as

$$v_{hq}^\mu \equiv \frac{p_{hq}^\mu}{m_Q} = v^\mu + \frac{k^\mu}{m_Q},$$

which is the same as that of the bound state in the heavy quark limit:

$$v_{hq}^\mu \rightarrow v^\mu \quad \text{as} \quad m_Q \rightarrow \infty.$$

When the heavy system is scattered by low-energy QCD interactions into a new state of the same heavy quark, the momentum conservation requires that

$$P^\mu = m_P v'^\mu + q'^\mu.$$

As $m_P \rightarrow \infty$ for a fixed momentum transfer q'^μ , we must have $v^\mu = v'^\mu$. That is, in the limit of infinite mass, the velocity of the heavy quark is unchanged by the QCD interactions, independently of what the light degrees of freedom do within its typical energy scale $\sim \Lambda_{QCD}$. Furthermore, two heavy quarks with different velocities do not communicate with each other, because their momenta are infinitely different and we stay at the energy scale of the light degrees of freedom. This “*velocity superselection rule*”⁴ makes it much easier to do physics with a heavy quark.

We first consider the dynamics of a free heavy quark in the effective theory. In the full theory, the heavy quark field h has the Lagrangian density

$$\mathcal{L}_{hq} = \bar{h}(x)(i\partial\!\!\!/ - m_Q)h(x), \quad (4.3)$$

in position space. For the heavy quark of velocity v , in evaluating the action written as an integral in momentum space as

$$\int \frac{d^4p}{(2\pi)^4} \bar{h}(-p)(\not{p} - m_Q)h(p), \quad (4.4)$$

with \not{p} denoting $a_\mu \gamma^\mu$, the relevant region of the integral is a cell of $p^\mu = m_Q v^\mu + k^\mu$ with $k^\mu \lesssim \Lambda_{QCD} \ll m_Q$ (see Fig. 4.1) and thus the relevant $h(p)$ is nearly on mass shell; $(\not{p} - m_Q)h(p) = O(1/m_Q) \approx 0$, which implies that

$$(\not{p} - 1)h(p) \approx -\frac{\not{k}}{m_Q}h(p) \approx 0.$$

All other quantities of order m_Q give a large contribution to the action and so make a small contribution to the integral. In a cell around $p^\mu = m_Q v^\mu$, define the heavy quark field $h_v(k) = h(p) - O(1/m_Q)$ satisfying exactly

$$\not{v}h_v = h_v. \quad (4.5)$$

Then, in terms of the residual momentum $k = p - m_Q v$, the Lagrangian in the cell looks like

$$\bar{h}_v(\not{p} - m_Q)h_v = \bar{h}_v\not{k}h_v = \frac{1}{2}\bar{h}_v\{\not{v}, \not{k}\}h_v = \bar{h}_v v^\mu k_\mu h_v.$$

Now, as $m_Q \rightarrow \infty$, the cell gets closer together in velocity space, but the size of each cell in momentum space grows. Each cell becomes a mini-Lagrangian relevant only for the heavy quark field with the corresponding velocity:

$$\int_{\text{cell}} \frac{d^4 k}{(2\pi)^4} \bar{c}_v(-k) v^\mu k_\mu c_v(k). \quad (4.6)$$

The velocity superselection rule is equivalent to the statement that h_v and $h_{v'}$ are independent fields for $v^\mu \neq v'^\mu$: they correspond to different cells on the mass shell hyperboloid. The Lagrangian is a sum over all v :

$$\mathcal{L}_{hq} = \sum_{\vec{v}} \mathcal{L}_{hq}[h_v]. \quad (4.7)$$

The corresponding Lagrangian for the heavy quark field in position space can be easily read off from Eq. (4.6)

$$\mathcal{L}_{hq}[h_v] = i\bar{h}_v(x) v \cdot \partial h_v(x). \quad (4.8)$$

The same Lagrangian could also be obtained by substituting⁴

$$h(x) = \frac{1 + \not{v}}{2} e^{-im_Q v \cdot x} h_v(x) + O(1/m_Q) \quad (4.9)$$

directly into Eq. (4.3). It projects away the negative light cone which describes anti-quarks and eliminates the trivial dependence on the heavy quark mass. With this effective Lagrangian, the Feynman propagator for the heavy quark field simplifies to

$$\frac{1 + \not{v}}{2} \frac{1}{v \cdot k + i\epsilon}, \quad (4.10)$$

which can be consistently derived by inserting $p^\mu = m_Q v^\mu + k^\mu$ into the Feynman propagator of the full theory and by keeping the leading order terms in m_Q in both the numerator and the denominator:

$$\frac{\not{p} + m_Q}{p^2 - m_Q^2 + i\epsilon} \approx \frac{m_Q \not{v} + m_Q}{2m_Q(v \cdot k) + i\epsilon} = \frac{1 + \not{v}}{2} \frac{1}{(v \cdot k) + i\epsilon}.$$

In the rest frame, we have

$$\frac{1 + \gamma^0}{2} \frac{1}{k^0 + i\epsilon},$$

whose Fourier transform is

$$\propto \delta^3(\vec{r} - \vec{r}') \Theta(t - t').$$

This just describes the particle sitting still, propagating in time along its classical trajectory.

QCD interactions can be easily incorporated by imposing the color gauge symmetry (we recall that the covariant derivative with the color gauge field was denoted D_μ^c):

$$\mathcal{L}_{hq}[h_v] = i\bar{h}_v v^\mu D_\mu^c h_v. \quad (4.11)$$

Here, the heavy-quark symmetry is manifest: the Lagrangian does not depend on the heavy quark mass (heavy-quark flavor symmetry) and does not contain any γ -matrices (heavy-quark spin symmetry); that is, by Eqs. (4.5) and (4.9) we have ignored the variation of the spinor within the cell and eliminated the trivial heavy quark mass dependence from the theory. Let us identify explicitly the symmetries of the heavy quark Lagrangian with two heavy flavors, say c and b :

$$\mathcal{L}_c[c_v] + \mathcal{L}_b[b_v] = i\bar{c}_v v^\mu D_\mu^c c_v + i\bar{b}_v v^\mu D_\mu^c b_v = i\bar{h}_v v^\mu D_\mu^c h_v, \quad (4.12)$$

where we have put the two fields together into an 8-component field:

$$h_v \equiv \begin{pmatrix} c_v \\ b_v \end{pmatrix}. \quad (4.13)$$

In the rest frame, $v^0=1$, $\vec{v}=0$, the Lagrangian is

$$\mathcal{L}_c[c_0] + \mathcal{L}_b[b_0] = i\bar{h}_0 D^{c0} h_0.$$

The Isgur-Wise heavy-quark symmetry is the $SU(4)$ spin-flavor symmetry characterized by the transformations whose generators are

$$P_0\sigma_j, \quad P_0\eta_j, \quad P_0\sigma_j\eta_k, \quad \text{for } j, k=1,2,3, \quad (4.14)$$

where P_0 and σ_j ($j=1,2,3$), respectively, are combinations of the 4×4 Dirac-matrices

$$P_0 \equiv \frac{1}{2}(1 + \gamma^0) \quad \text{and} \quad \vec{\sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix},$$

and η_j ($j=1,2,3$) is an additional set of Pauli matrices that implements the $SU(2)$ rotation between c and b subspaces.

4.2. Heavy Meson Theory

Returning to the Lagrangian (2.19) for the heavy mesons, we proceed in a way analogous to the heavy-quark Lagrangian. First, in analogy to Eq. (4.9), we introduce new heavy meson fields for each velocity:

$$\begin{aligned} P &= e^{-iv \cdot x \bar{m}_P} \frac{1}{\sqrt{2\bar{m}_P}} P_v, \\ P_\mu^* &= e^{-iv \cdot x \bar{m}_P} \frac{1}{\sqrt{2\bar{m}_P}} P_{v\mu}^*, \end{aligned} \quad (4.15)$$

where $\bar{m} \equiv \frac{1}{4}(m_P + 3m_{P^*})$ is the averaged mass of the heavy mesons. Note that in the infinite m_Q limit we have $m_P = m_{P^*}$. The factor $1/\sqrt{2\bar{m}_P}$ is introduced for later convenience. Substituting Eq. (4.15) into the Lagrangian (2.19) and keeping the leading order terms in \bar{m}_P , we obtain a heavy-flavor independent Lagrangian

$$\begin{aligned}\mathcal{L}_v = \mathcal{L}_M + iv^\mu(D_\mu P_v P_v^\dagger - D_\mu P_{v\nu}^* P_{v\nu}^{*\dagger}) \\ + \frac{1}{2}g(P_v A^\mu P_{v\mu}^{*\dagger} + P_{v\mu}^* A^\mu P_v^\dagger) - i\frac{1}{2}g\varepsilon^{\mu\nu\lambda\rho}v_\mu P_{v\nu}^* A_\lambda P_{v\rho}^{*\dagger}.\end{aligned}\quad (4.16)$$

Here, we have used Eq. (2.36) for the coupling constants in the heavy quark limit and the constraint that $v \cdot P_v^* = 0$, and replaced $P_v D_\mu P_v^\dagger$ by $-D_\mu P_v P_v^\dagger$, dropping the trivial total derivative term $\partial_\mu(P_v P_v^\dagger)$.

A special care should be taken in this procedure. The Lagrangian (2.19) defines the canonical momenta conjugate to the fields P , P^\dagger , P^{*i} and $P^{*i\dagger}(i=1,2,3)$ ^{#7}:

$$\begin{aligned}\Pi^\dagger &\equiv \frac{\delta\mathcal{L}}{\delta\dot{P}} = D_0 P^\dagger, & \Pi &\equiv \frac{\delta\mathcal{L}}{\delta\dot{P}^\dagger} = D_0 P, \\ \Pi^{*i\dagger} &\equiv \frac{\delta\mathcal{L}}{\delta\dot{P}^{*i}} = P^{*0i\dagger}, & \Pi^{*i} &\equiv \frac{\delta\mathcal{L}}{\delta\dot{P}^{*i\dagger}} = P^{*0i}\end{aligned}\quad (4.17)$$

where \dot{a} denotes $\partial_0 a$. Their equal-time commutation relations are

$$\begin{aligned}[P_{i_3}(t, \vec{r}), \Pi_{i'_3}^\dagger(t, \vec{r}')] &= [P_{i_3}^\dagger(t, \vec{r}), \Pi_{i'_3}(t, \vec{r}')] = i\delta_{i_3 i'_3} \delta^3(\vec{r} - \vec{r}'), \\ [P_{i_3}^{*j}(t, \vec{r}), \Pi_{i'_3}^{*k\dagger}(t, \vec{r}')] &= [P_{i_3}^{*j\dagger}(t, \vec{r}), \Pi_{i'_3}^{*k}(t, \vec{r}')] = i\delta^{jk} \delta_{i_3 i'_3} \delta^3(\vec{r} - \vec{r}'),\end{aligned}\quad (4.18)$$

where the subscript $i_3(i'_3)$ is introduced to denote explicitly the isospin of the fields. On the other hand, the Lagrangian (4.16) defines the canonical momenta conjugate to the fields P_v and P_v^{*i}

$$\begin{aligned}\Pi_v^\dagger &\equiv \frac{\delta\mathcal{L}_v}{\delta\dot{P}_v} = iv^0 P_v^\dagger, \\ \Pi_v^{*i\dagger} &\equiv \frac{\delta\mathcal{L}_v}{\delta\dot{P}_v^{*i}} = iv^0 P_v^{*i\dagger}.\end{aligned}\quad (4.19)$$

Their commutation relations

$$\begin{aligned}[P_{vi_3}(t, \vec{r}), \Pi_{vi'_3}^\dagger(t, \vec{r}')] &= [P_{vi_3}(t, \vec{r}), iv^0 P_{vi_3}^\dagger(t, \vec{r}')] = i\delta_{i_3 i'_3} \delta^3(\vec{r} - \vec{r}'), \\ [P_{vi_3}^{*j}(t, \vec{r}), \Pi_{vi'_3}^{*k\dagger}(t, \vec{r}')] &= [P_{vi_3}^{*j}(t, \vec{r}), iv^0 P_{vi'_3}^{*k\dagger}(t, \vec{r}')] = i\delta^{jk} \delta_{i_3 i'_3} \delta^3(\vec{r} - \vec{r}'),\end{aligned}\quad (4.20)$$

appear to be inconsistent with what we would have obtained by inserting Eq. (4.15) naively into Eq. (4.18) with (4.17), which results in commutation relations that differ by a factor of 2.

To understand what is going on, consider the free pseudoscalar meson field. (The same is true for vector mesons as well as for interacting fields.) The meson field operator P can be expanded in terms of the classical eigenmodes (plane wave solutions) as

$$P(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2\omega_p)}} (e^{-ip \cdot x} a(p) + e^{ip \cdot x} b^\dagger(p)), \quad (4.21)$$

^{#7} P^{*0} and $P^{*0\dagger}$ are not dynamical variables and do not have conjugate momenta.

where $\omega_p = \sqrt{m_P^2 + \vec{p}^2}$ and $a(p)$ [$b^\dagger(p)$] is the annihilation (creation) operator of heavy meson (its anti-particle) which satisfies the commutation relations

$$[a(p), a^\dagger(p')] = [b(p), b^\dagger(p')] = \delta^3(\vec{p} - \vec{p}'), \quad (4.22)$$

all the other commutators vanish.

For simplicity, we are ignoring here the isospin structure of the meson field. In the cell around $p^\mu = m_P v^\mu$ of our interest, the anti-particles are projected out by Eq. (4.15) and hence the field operator P_v is expanded only in terms of the meson particle states,

$$P_v(x) = \int \frac{d^3k}{\sqrt{v^0(2\pi)^3}} e^{-ik \cdot x} a(k). \quad (4.23)$$

The integral is over the cell $k^\mu \lesssim \Lambda_{QCD} \ll m_P \approx m_Q$ and we have used that $\omega_p \approx m_P v^0$ in that region. This simple exercise explains the factor-2 problem in the commutation relations. In using Eq. (4.15), one has to keep in mind that it eliminates the anti-particles from the theory. Note that the plane wave solutions in Eq. (4.23), $\varphi_{v\vec{k}}(\vec{r}, t) \equiv [v^0(2\pi)^3]^{-1/2} e^{-ik \cdot x}$, are normalized as

$$v_0 \int d^3r (\varphi_{v\vec{k}}^* \varphi_{v\vec{k}'}) = \delta^3(\vec{k} - \vec{k}'),$$

while those in Eq. (4.21), $\varphi_{\vec{p}}(\vec{r}, t) \equiv [(2\pi)^3 2\omega_p]^{-1/2} e^{-ip \cdot x}$, are normalized as

$$i \int d^3r (\varphi_{\vec{p}}^* \dot{\varphi}_{\vec{p}'} - \dot{\varphi}_{\vec{p}}^* \varphi_{\vec{p}'}) = \delta^3(\vec{p} - \vec{p}').$$

In the rest frame, $v=(1,0,0,0)$, the Lagrangian (4.16) is further simplified to

$$\mathcal{L}_0 = \mathcal{L}_M + i(D_0 P_v P_v^\dagger + D_0 \vec{P}_v^* \cdot \vec{P}_v^{\dagger}) - g(P_v \vec{A} \cdot \vec{P}_v^{\dagger} + \vec{P}_v^* \cdot \vec{A} P_v^\dagger) - ig \vec{P}_v^* \cdot \vec{A} \times \vec{P}_v^{\dagger}. \quad (4.24)$$

One can easily identify the heavy-quark spin symmetry with this Lagrangian. In analogy with Eq. (2.31), we rewrite P_v and P_v^{*i} ($i=1,2,3$)^{#8} as combinations of the four fields $P_{s_Q s_\ell}$ ($s_Q, s_\ell = \uparrow, \downarrow$) as

$$\begin{aligned} P_v &= \frac{1}{\sqrt{2}}(P_{\uparrow\downarrow} - P_{\downarrow\uparrow}), \\ P_v^{*1} &= \frac{1}{\sqrt{2}}(P_{\uparrow\uparrow} - P_{\downarrow\downarrow}), \\ P_v^{*2} &= \frac{i}{\sqrt{2}}(P_{\uparrow\uparrow} + P_{\downarrow\downarrow}), \\ P_v^{*3} &= \frac{1}{\sqrt{2}}(P_{\uparrow\downarrow} + P_{\downarrow\uparrow}), \end{aligned} \quad (4.25)$$

where s_Q and s_ℓ denote the heavy quark spin and the total angular momentum of the light degrees of freedom, respectively. Then, Lagrangian (4.24) can be written in terms of these new fields,

$$\mathcal{L}_0 = \mathcal{L}_M + \mathcal{L}_{0\uparrow} + \mathcal{L}_{0\downarrow}, \quad (4.26)$$

^{#8}In the rest frame P_v^{*0} is identically zero due to the condition that $v \cdot P_v^* = 0$.

where

$$\begin{aligned}\mathcal{L}_{0\uparrow} = & i(D_0 P_{\uparrow\uparrow} P_{\uparrow\uparrow}^\dagger + D_0 P_{\uparrow\downarrow} P_{\uparrow\downarrow}^\dagger) \\ & + g(P_{\uparrow\uparrow} A^0 P_{\uparrow\uparrow}^\dagger + P_{\uparrow\uparrow} A^+ P_{\uparrow\downarrow}^\dagger + P_{\uparrow\downarrow} A^- P_{\uparrow\uparrow}^\dagger - P_{\uparrow\downarrow} A^0 P_{\uparrow\downarrow}^\dagger),\end{aligned}\quad (4.26a)$$

and

$$\begin{aligned}\mathcal{L}_{0\downarrow} = & i(D_0 P_{\downarrow\uparrow} P_{\downarrow\uparrow}^\dagger + D_0 P_{\downarrow\downarrow} P_{\downarrow\downarrow}^\dagger) \\ & + g(P_{\downarrow\uparrow} A^0 P_{\downarrow\uparrow}^\dagger + P_{\downarrow\uparrow} A^+ P_{\downarrow\downarrow}^\dagger + P_{\downarrow\downarrow} A^- P_{\downarrow\uparrow}^\dagger - P_{\downarrow\downarrow} A^0 P_{\downarrow\downarrow}^\dagger),\end{aligned}\quad (4.26b)$$

with $A^\pm \equiv (A^1 \pm iA^2)$ and $A^0 \equiv A^3$. Here $\mathcal{L}_{0\uparrow}$ contains only the fields with heavy quark spin up and $\mathcal{L}_{0\downarrow}$ only those with spin down. Then the Lagrangian is completely separated into two identical copies for each heavy quark spin s_h , so it is symmetric under the transformations

$$\begin{pmatrix} P_{\uparrow s_\ell} \\ P_{\downarrow s_\ell} \end{pmatrix} \rightarrow \begin{pmatrix} P'_{\uparrow s_\ell} \\ P'_{\downarrow s_\ell} \end{pmatrix} = i\vec{\epsilon} \cdot \vec{\sigma} \begin{pmatrix} P_{\uparrow s_\ell} \\ P_{\downarrow s_\ell} \end{pmatrix}, \quad (s_\ell = \uparrow \text{ and } \downarrow) \quad (4.27)$$

with a set of Pauli matrices $\vec{\sigma}$ acting on the spinor fields $\begin{pmatrix} P_{\uparrow s_\ell} \\ P_{\downarrow s_\ell} \end{pmatrix}$ ($s_\ell = \uparrow$ and \downarrow).

In terms of the field operators $P_{s_h s_\ell}$, the corresponding heavy-quark spin operators \vec{S}_Q are defined by

$$\begin{aligned}S_Q^3 &= \frac{1}{2} \int d^3r (P_{\uparrow\uparrow} P_{\uparrow\uparrow}^\dagger + P_{\uparrow\downarrow} P_{\uparrow\downarrow}^\dagger - P_{\downarrow\uparrow} P_{\downarrow\uparrow}^\dagger - P_{\downarrow\downarrow} P_{\downarrow\downarrow}^\dagger), \\ S_Q^+ &\equiv S_h^1 + iS_h^2 = - \int d^3r (P_{\downarrow\uparrow} P_{\uparrow\uparrow}^\dagger + P_{\downarrow\downarrow} P_{\uparrow\downarrow}^\dagger), \\ S_Q^- &\equiv S_h^1 - iS_h^2 = - \int d^3r (P_{\uparrow\uparrow} P_{\downarrow\uparrow}^\dagger + P_{\uparrow\downarrow} P_{\downarrow\downarrow}^\dagger).\end{aligned}\quad (4.28)$$

Using the equal time commutation relations for the field operators $P_{s_Q s_\ell}^{\#9}$, one can easily show that they satisfy the correct $SU(2)$ spin algebra

$$[S_Q^i, S_Q^j] = i\epsilon^{ijk} S_Q^k, \quad [S_Q^3, S_Q^\pm] = \pm S_Q^\pm, \quad (4.28a)$$

and the correct commutation relations with the fields; *e.g.*,

$$[S_Q^3, P_{\uparrow\uparrow}] = -\frac{1}{2} P_{\uparrow\uparrow}, \quad [S_Q^+, P_{\uparrow\uparrow}] = P_{\downarrow\uparrow}, \quad \dots \quad (4.28b)$$

These relations might appear strange. However they represent correct spin operators; one should keep in mind that $P_{\uparrow\uparrow}$ is the *annihilation operator* of the $s_Q = +\frac{1}{2}$ and $s_\ell = +\frac{1}{2}$ particle. Rewriting the heavy quark spin operators in terms of P_v and P_v^{*i} , we have

$$\vec{S}_Q = \frac{1}{2} \int d^3r [i(\vec{P}_v^* \times \vec{P}_v^\dagger) + (P_v \vec{P}_v^* - \vec{P}_v^* P_v^\dagger)]. \quad (4.29)$$

^{#9}They are

$$[P_{s_Q s_\ell i_3}(t, \vec{r}), P_{s'_Q s'_\ell i'_3}^\dagger(t, \vec{r}')] = \delta_{s_Q s'_Q} \delta_{s_\ell s'_\ell} \delta_{i_3 i'_3} \delta^3(\vec{r} - \vec{r}'),$$

in the rest frame ($v_0=1$).

Note that they are not simply one-half of the spin operators of the heavy meson field^{#10}

$$\vec{S} = i \int d^3r \vec{P}_v^* \times \vec{P}_v^{\dagger}. \quad (4.30)$$

The spin of the light degrees of freedom, \vec{S}_ℓ , can be defined by $\vec{S}_\ell = \vec{S} - \vec{S}_Q$: *viz.*

$$\vec{S}_\ell = \frac{1}{2} \int d^3r [i(\vec{P}_v^* \times \vec{P}_v^{\dagger}) - (P_v \vec{P}_v^{\dagger} - \vec{P}_v^* P_v^{\dagger})]. \quad (4.31)$$

To be consistent, they can be rewritten in terms of the fields operator $P_{s_h s_\ell}$ as

$$\begin{aligned} S_\ell^3 &= \frac{1}{2} \int d^3r (P_{\uparrow\uparrow} P_{\uparrow\uparrow}^\dagger - P_{\uparrow\downarrow} P_{\uparrow\downarrow}^\dagger + P_{\downarrow\uparrow} P_{\downarrow\uparrow}^\dagger - P_{\downarrow\downarrow} P_{\downarrow\downarrow}^\dagger), \\ S_\ell^+ &\equiv S_\ell^1 + i S_\ell^2 = - \int d^3r (P_{\uparrow\downarrow} P_{\uparrow\uparrow}^\dagger + P_{\downarrow\downarrow} P_{\downarrow\uparrow}^\dagger), \\ S_\ell^- &\equiv S_\ell^1 - i S_\ell^2 = - \int d^3r (P_{\uparrow\uparrow} P_{\uparrow\downarrow}^\dagger + P_{\downarrow\uparrow} P_{\downarrow\downarrow}^\dagger). \end{aligned}$$

Such expressions can be written in a compact form by taking a 4×4 matrix representation.^{5,38,39} Let $H(x)$ be a 4×4 matrix field defined by

$$H(x) = \frac{1 + \psi}{2} (\gamma_5 P_v - \gamma_\mu P_v^{*\mu}), \quad (4.32)$$

^{#10}The Lagrangian (2.19) is invariant under the infinitesimal Lorentz transformation

$$\begin{aligned} x^\mu &\rightarrow x^{\mu'} = x^\mu + \epsilon^\mu{}_\nu x^\nu, \text{ with } \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \\ P(x) &\rightarrow P'(x') = P(x), \\ P_\alpha^*(x) &\rightarrow P_\alpha^{*'}(x') = P_\alpha^*(x) + \epsilon_\alpha{}^\beta P_\beta^*(x) = P_\alpha^*(x) + \frac{1}{2} \epsilon^{\mu\nu} (S_{\mu\nu})_\alpha{}^\beta P_\beta^*(x), \end{aligned}$$

with

$$(S^{\mu\nu})_{\alpha\beta} = g^\mu{}_\alpha g^\nu{}_\beta - g^\mu{}_\beta g^\nu{}_\alpha$$

and gives a conserved angular momentum

$$M_{\rho\sigma} = \int d^3r \mathcal{M}_{0\rho\sigma},$$

where

$$\begin{aligned} \mathcal{M}^{\mu\rho\sigma} &= (x^\rho \mathcal{T}^{\mu\sigma} - x^\sigma \mathcal{T}^{\mu\rho}) \\ &\quad + P^{*\beta\dagger} (S^{\sigma\rho})_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial (\partial_\mu P_\alpha^{*\dagger})} + \frac{\partial \mathcal{L}}{\partial (\partial_\mu P_\alpha^{*\dagger})} (S^{\sigma\rho})_{\alpha\beta} P^{*\beta\dagger} \end{aligned}$$

with the canonical energy-momentum tensor $\mathcal{T}^{\mu\nu}$. The indices ρ and σ run from 1 to 3. [The operators with ρ (or σ)=0 are for the Lorentz boost, in which case the second term vanishes due to $\partial \mathcal{L} / \partial \dot{P}_0^* = 0$.] The first part is the orbital angular momentum carried by the P and P^* mesons and the second part the spin angular momentum carried by P^* mesons. Explicitly, the spin angular momentum can be written as

$$\begin{aligned} S^i &\equiv \frac{1}{2} \varepsilon^{ijk} S^{jk} = -\frac{1}{2} \varepsilon^{ijk} \int d^3r (\Pi^{*\alpha} (S^{jk})_{\alpha\beta} P^{*\beta\dagger} + P^{*\beta} (S^{jk})_{\alpha\beta} \Pi^{*\alpha\dagger}) \\ &= -\varepsilon^{ijk} \int d^3r (\Pi^{*j} P^{*k\dagger} + P^{*k} \Pi^{*j\dagger}). \end{aligned}$$

In the limit $m_{P^*} \rightarrow \infty$, substitution of Eq. (4.15) results in Eq. (4.30) as the leading order term in meson mass.

with the conventional Dirac γ -matrices. For example, the Lagrangian density (4.16) can be rewritten simply as³⁶

$$\mathcal{L}_v = \mathcal{L}_M - iv_\mu \text{Tr}(D^\mu H \bar{H}) - g \text{Tr}(H \gamma_5 A_\mu \gamma^\mu \bar{H}), \quad (4.33)$$

where

$$\bar{H} = \gamma_0 H^\dagger \gamma_0. \quad (4.33a)$$

The parity transformations (2.23) are

$$\mathcal{P} H(\vec{r}, t) \mathcal{P}^{-1} = \gamma^0 H(-\vec{r}, t) \gamma^0. \quad (4.33b)$$

Explicitly, in the rest frame, $H(x)$ has the structure

$$H(x) = \begin{bmatrix} 0 & 0 & +P_{\downarrow\downarrow} + P_{\uparrow\uparrow} \\ 0 & 0 & -P_{\downarrow\downarrow} - P_{\downarrow\uparrow} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4.33c)$$

that is,

$$H_{13} = +P_{\downarrow\downarrow}, \quad H_{14} = +P_{\uparrow\uparrow}, \quad H_{23} = -P_{\downarrow\downarrow}, \quad H_{24} = -P_{\downarrow\uparrow}, \quad (4.33d)$$

and all the other 12-components vanish. The spin operators can be summarized as

$$\begin{aligned} \vec{S} &= -\int d^3r \text{Tr}([\tfrac{1}{2}\vec{\sigma}, H] \bar{H}), \\ \vec{S}_Q &= -\int d^3r \text{Tr}(\tfrac{1}{2}\vec{\sigma} H \bar{H}), \\ \vec{S}_\ell &= +\int d^3r \text{Tr}(H \tfrac{1}{2}\vec{\sigma} \bar{H}), \end{aligned} \quad (4.34)$$

with the 4×4 spin matrices defined by

$$\sigma^i \equiv \tfrac{1}{4} i \varepsilon^{ijk} [\gamma^j, \gamma^k].$$

The Hamiltonian of the system can be obtained by taking the Legendre transform of Lagrangian (4.33):

$$\begin{aligned} H &= \int d^3r \left\{ \left(\frac{\delta \mathcal{L}_v}{\delta \dot{U}_{\iota\kappa}} \dot{U}_{\iota\kappa} + \text{h.c.} \right) + \frac{\delta \mathcal{L}_v}{\delta \dot{H}_{\rho\sigma}} \dot{H}_{\rho\sigma} - \mathcal{L}_v \right\} \\ &= \int d^3r \left\{ \left(\tfrac{1}{4} f_\pi^2 \text{Tr}(\dot{U}^\dagger \dot{U} + \vec{\nabla} U^\dagger \cdot \vec{\nabla} U) + \dots \right) \right. \\ &\quad \left. - i \vec{v} \cdot \text{Tr}(\vec{D} H \bar{H}) + g \text{Tr}(H \gamma_5 A_\mu \gamma^\mu \bar{H}) \right\}, \end{aligned} \quad (4.35)$$

where ι and κ run from 1 to 2 for the 2×2 matrix U , and ρ and σ run from 1 to 4 for the 4×4 matrix H .

We derived the above Lagrangian (4.33) starting from a traditional meson Lagrangian by using the heavy quark symmetry. Originally, it was constructed by Wise³⁶ directly from the heavy quark symmetry.

5. Heavy Baryons as Skyrmions

5.1. Heavy-Meson-Soliton Bound State

The non-linear Lagrangian \mathcal{L}_M supports a classical soliton solution

$$U_0(\vec{r}) = \exp[i\vec{\tau} \cdot \hat{r}F(r)], \quad (5.1)$$

with the boundary conditions

$$F(0) = \pi \quad \text{and} \quad F(r) \xrightarrow{r \rightarrow \infty} 0, \quad (5.1a)$$

which, due to its nontrivial topological structure, carries a winding number identified as the baryon number

$$\begin{aligned} B &= -\frac{1}{24\pi^2} \int d^3r \, \varepsilon^{ijk} \text{Tr}(U_0^\dagger \partial_i U_0 U_0^\dagger \partial_j U_0 U_0^\dagger \partial_k U_0) \\ &= -\frac{2}{\pi} \int_0^\infty r^2 dr \frac{\sin^2 F}{r^2} F' = 1, \end{aligned} \quad (5.1b)$$

and a finite mass

$$M_{sol} = 4\pi \int_0^\infty r^2 dr \frac{f_\pi^2}{2} \left(F'^2 + 2 \frac{\sin^2 F}{r^2} \right), \quad (5.1c)$$

with $F' = dF/dr$. With \mathcal{L}_M alone, however, the solution is unstable against scale transformations; under the transformation $U(\vec{r}) \rightarrow U(\lambda\vec{r})$, the mass scales as $M_{sol} \rightarrow \lambda^{-1}M_{sol}$. That is, it collapses into a pointlike and zero-energy configuration. The stabilization can be established simply by adding a quartic term (3.17) to the Lagrangian, which scales like r^{-4} or by incorporating the repulsion mediated by vector mesons at short distance. As mentioned above, the Skyrme quartic term can be considered as the infinite mass limit of the ρ exchange. (See Sec. 3) In Fig. 5.1 is presented a typical wave function $F(r)$ stabilized by the Skyrme term.^{72,73}

The next step is to find the eigenstates of the heavy meson fields interacting with the static potentials

$$\begin{aligned} V^\mu &= (V^0, \vec{V}) = (0, -i(\vec{\tau} \times \hat{r}) \frac{\sin^2(F/2)}{r}), \\ A^\mu &= (A^0, \vec{A}) = (0, \frac{1}{2}[\frac{\sin F}{r}\vec{\tau} + (F' - \frac{\sin F}{r})\hat{r}\vec{\tau} \cdot \hat{r}]), \end{aligned} \quad (5.2)$$

given by the $B=1$ soliton configuration (5.1) sitting at the origin, with a focus on meson-soliton bound state(s). In the rest frame, the Lagrangian takes the form

$$L_0 = -M_{sol} + \int d^3r \left\{ -i \text{Tr}(\partial_0 H \bar{H}) + g \text{Tr}(H \vec{A} \cdot \vec{\sigma} \bar{H}) \right\}, \quad (5.3)$$

from which follows the equation of motion

$$i\partial_0 h(\vec{r}, t) = \varepsilon h(\vec{r}, t) = h(\vec{r}, t)[g\vec{A} \cdot \vec{\sigma}], \quad (5.4)$$

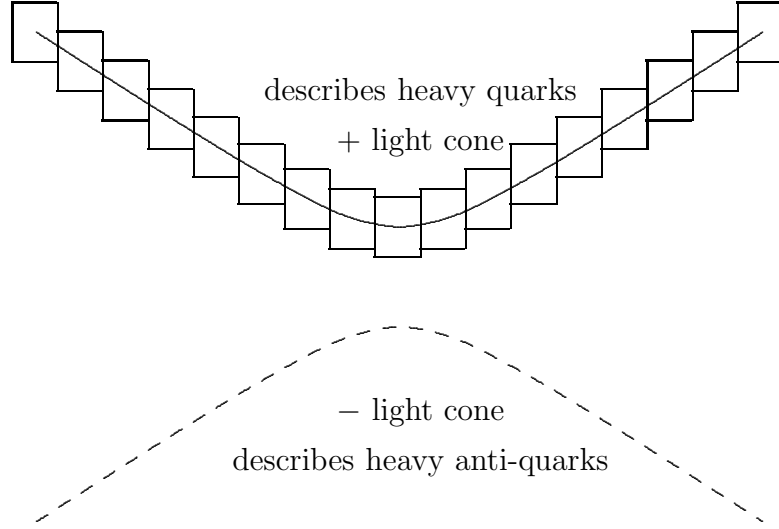


Figure 4.1: Momentum space for the functional integral of eq.(4.4). The + light cone is divided up into cells of width $\sqrt{m_Q\Lambda}$, *i.e.* $\Delta v \approx \sqrt{\Lambda/m_Q}$.

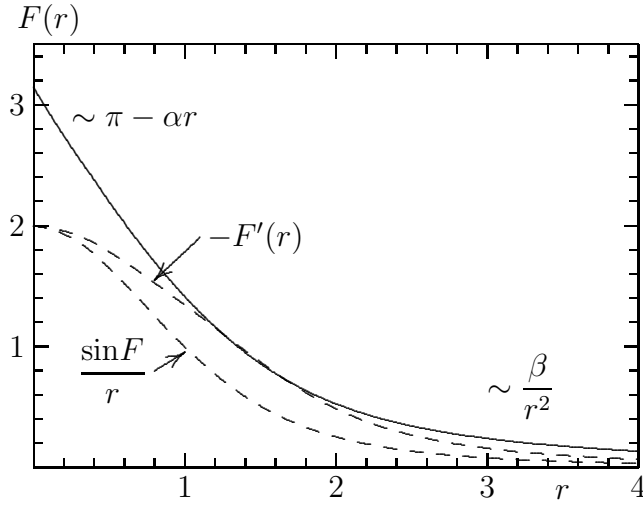


Figure 5.1: Typical hedgehog profile function $F(r)$ (stabilized by the Skyrme term, r in unit of $(ef_\pi)^{-1}$).

for the wavefunction $h(\vec{r}, t)$ of the classical eigenmode with an eigenenergy ε .

In the “hedgehog” configuration (5.1) – and consequently in the static potentials (5.2), the isospin and the angular momentum are correlated in such a way that neither of them is separately a good quantum number, but their sum (the grand spin) is;

$$\vec{K} = \vec{J} + \vec{I} \equiv (\vec{L} + \vec{S}) + \vec{I}. \quad (5.5)$$

Thus, the equation of motion (5.4) is invariant under rotations in K -space and the eigenstates are classified by the quantum numbers $(k; k_3)$.^{#11} The wavefunctions of the heavy meson eigenmodes can be written as the product of a radial function $h(r)$ and the eigenfunction of the grand spin $\mathcal{K}_{kk_3}(\hat{r})$:

$$h(\vec{r}, t) = \sum_i \alpha_i h_k^i(r) \mathcal{K}_{kk_3}^i(\hat{r}) e^{-i\varepsilon t}, \quad (5.6)$$

where the sum is over the possible ways of constructing the eigenstates of the same grand spin and parity by combining the eigenstates of the spin, isospin and orbital angular momentum, and the expansion coefficients α_i are normalized by $\sum_i |\alpha_i|^2 = 1$. We assume that both the soliton and the heavy mesons are infinitely heavy so that in the lowest energy state they are on top of each other at the same spatial point,^{#12} just propagating in time. That is, the radial functions $h_k^i(r)$ of the lowest energy eigenstate can be approximated by a delta-function-like one, say $h(r)$, which is strongly peaked at the origin and normalized as $\int r^2 dr |h(r)|^2 = 1$. Thus, the wavefunction is normalized as

$$- \int d^3r \text{Tr}(h\bar{h}) = 1, \quad (5.7)$$

with the orthonormalized eigenfunction $\mathcal{K}_{kk_3}^i(\hat{r})$ of the grand spin

$$\int d\Omega \text{Tr} \left(\mathcal{K}_{kk_3}^i(\hat{r}) \bar{\mathcal{K}}_{k'k'_3}^{i'}(\hat{r}) \right) = -\delta_{ii'} \delta_{kk'} \delta_{k_3k'_3}. \quad (5.8)$$

By integrating out the radial part of the equation of motion (5.4), we obtain

$$\varepsilon \mathcal{K}_{kk_3}(\hat{r}) = \mathcal{K}_{kk_3}(\hat{r}) \left[\frac{1}{2} g F'(0) (2\vec{\sigma} \cdot \hat{r} \vec{\tau} \cdot \hat{r} - \vec{\sigma} \cdot \vec{\tau}) \right], \quad (5.9)$$

with $\mathcal{K}_{kk_3} \equiv \sum_i c_i \mathcal{K}_{kk_3}^i$. Here, we have used that $F(r) \sim \pi + F'(0)r$ and consequently $\vec{A} \cdot \vec{\sigma} \sim \frac{1}{2} F'(0) (2\vec{\sigma} \cdot \hat{r} \vec{\tau} \cdot \hat{r} - \vec{\sigma} \cdot \vec{\tau})$ near the origin. Now, our problem is reduced to

^{#11}More specifically, the grand spin operator can be written as

$$\vec{K} = \vec{L} + \vec{S}_\ell + \vec{S}_h + \vec{I} \equiv \vec{K}_\ell + \vec{S}_h,$$

and the heavy quark spin symmetry implies that the heavy quark spin decouples. That is, the equation of motion is invariant separately under the heavy quark spin rotations and under rotations by \vec{K}_ℓ , the grand spin of the light degrees of freedom. For comparison with the conventional bound-state approach¹² in the Skyrme model, we will work with the grand spin \vec{K} (not the grand spin of the light degrees of freedom, \vec{K}_ℓ).

^{#12}Note that the potentials take their minimum values at the origin. (See Fig. 5.1.)

finding \mathcal{K}_{kk_3} . First, we construct the grand spin eigenstates $\mathcal{K}_{kk_3}^i(\hat{r})$ by combining the eigenstates of the spin, isospin and orbital angular momentum.

Equation (4.34) implies that the quantum mechanical spin operators acting on the wavefunctions are of the form

$$\begin{aligned}\vec{S}\{h\} &= [\tfrac{1}{2}\vec{\sigma}, h] \quad \text{spin of the heavy-meson,} \\ \vec{S}_h\{h\} &= \tfrac{1}{2}\vec{\sigma}h \quad \text{heavy-quark spin,} \\ \vec{S}_\ell\{h\} &= -h\tfrac{1}{2}\vec{\sigma} \quad \text{spin of the brown muck.}\end{aligned}\tag{5.10}$$

The minus signs in Eq. (4.34) originate from the normalization of the H -field. Note that the heavy *flavor* number operator

$$N_f = 2f_h \int d^3r (P_v P_v^\dagger + \vec{P}_v^* \cdot \vec{P}_v^{*\dagger}),$$

with the heavy quark flavor $f_h (f_c \equiv +1, f_b \equiv -1)$ is expressed in terms of H field as

$$N_f = -f_h \int d^3r \text{Tr}(H\bar{H}).$$

Since we are working with the heavy mesons of spin 0 and 1, the corresponding eigenstates $(s; s_3)$ of the spin operator \vec{S} can be readily written down,

$$\begin{aligned}(0; 0) &= [\tfrac{1}{2\sqrt{2}}(1 + \gamma^0)]\gamma_5, \\ (1, +1) &= [\tfrac{1}{2\sqrt{2}}(1 + \gamma^0)][\tfrac{-1}{\sqrt{2}}(\gamma^1 + i\gamma^2)], \\ (1; 0) &= [\tfrac{1}{2\sqrt{2}}(1 + \gamma^0)]\gamma^3, \\ (1, -1) &= [\tfrac{1}{2\sqrt{2}}(1 + \gamma^0)][\tfrac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2)],\end{aligned}\tag{5.11}$$

with the states normalized as

$$-\text{Tr}\{(s; s_3)\overline{(s'; s'_3)}\} = \delta_{ss'}\delta_{s_3s'_3}.\tag{5.11a}$$

The overall factor $[\tfrac{1}{2}(1 + \gamma^0)]$ is to incorporate the fact that these states are for the heavy mesons (not for their anti-particles). The explicit eigenstates of the spin operators \vec{S}_h and \vec{S}_ℓ are listed in Table 5.1. The relative phase between the three states $(s = 1; s_3)$ and the one between the four states $(s_h = \tfrac{1}{2}; \pm\tfrac{1}{2}|s_\ell = \tfrac{1}{2}; \pm\tfrac{1}{2})$ are fixed by the angular momentum structure; *i.e.*, they satisfy

$$\begin{aligned}[\tfrac{1}{2}\sigma_\pm, (s = 1; \mp 1)] &= \sqrt{2}(s = 1; 0), \\ \tfrac{1}{2}\sigma_\pm(s_h = \tfrac{1}{2}; \mp\tfrac{1}{2}|s_\ell; s_{\ell 3}) &= (s_h = \tfrac{1}{2}; \pm\tfrac{1}{2}|s_\ell; s_{\ell 3}), \\ (s_h = \tfrac{1}{2}; s_{h3}|s_\ell; \mp\tfrac{1}{2})(-\tfrac{1}{2}\sigma_\pm) &= (s_h = \tfrac{1}{2}; s_{h3}|s_\ell; \pm\tfrac{1}{2}).\end{aligned}\tag{5.11c}$$

Table 5.1 : Eigenstates of heavy-quark spin operators \vec{S}_h and the spin operator of the light degrees of freedom \vec{S}_ℓ .

$(s_h = \frac{1}{2}; +\frac{1}{2} s_\ell = \frac{1}{2}; +\frac{1}{2})$	$= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)] [\frac{-1}{\sqrt{2}}(\gamma^1 + i\gamma^2)]$
$(s_h = \frac{1}{2}; +\frac{1}{2} s_\ell = \frac{1}{2}; -\frac{1}{2})$	$= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)] [\frac{1}{\sqrt{2}}(\gamma^3 + \gamma^5)]$
$(s_h = \frac{1}{2}; -\frac{1}{2} s_\ell = \frac{1}{2}; +\frac{1}{2})$	$= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)] [\frac{1}{\sqrt{2}}(\gamma^3 - \gamma^5)]$
$(s_h = \frac{1}{2}; -\frac{1}{2} s_\ell = \frac{1}{2}; -\frac{1}{2})$	$= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)] [\frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2)]$

However, one may still choose an arbitrary phase for the state (0;0) without altering the final results.^{#13}

The invariance of the Lagrangian (2.19) under the isovector transformations (2.14) and (2.17) with $L = R = \vartheta = V$ gives us the isospin operator

$$\vec{I} = \vec{I}_M + \vec{I}_h, \quad (5.12)$$

with \vec{I}_M the isospin operator acting on the Goldstone boson fields

$$\vec{I}_M = i \int d^3r \frac{f_\pi^2}{2} \text{Tr} \{ \frac{1}{2} \vec{\tau} (U^\dagger \partial_0 U + U \partial_0 U^\dagger) \}, \quad (5.12a)$$

and \vec{I}_h on the heavy meson fields

$$\vec{I}_h = i \int d^3r \left\{ \Pi \left[-\frac{1}{4} (\xi^\dagger \vec{\tau} \xi + \xi \vec{\tau} \xi^\dagger) \right] P^\dagger + \Pi^{*i} \left[-\frac{1}{4} (\xi^\dagger \vec{\tau} \xi + \xi \vec{\tau} \xi^\dagger) \right] P^{*i\dagger} + (\text{h.c.}) \right\}. \quad (5.12b)$$

Note that the covariant couplings to the Goldstone bosons contribute to the isospin operator. The isospin operator of the free heavy mesons can be obtained by turning off the couplings; *i.e.*, $\xi = 1$.

^{#13}The eigenstate of the (0;0) state, $\frac{1}{2\sqrt{2}}(1 + \gamma^0)\gamma_5$, corresponds to the following convention for addition of two spins in non-relativistic quantum mechanics:

$$\begin{aligned} (s = 0; 0) &= \frac{1}{\sqrt{2}} [(s_h = \frac{1}{2}; +\frac{1}{2} | s_\ell = \frac{1}{2}; -\frac{1}{2}) - (s_h = \frac{1}{2}; -\frac{1}{2} | s_\ell = \frac{1}{2}; +\frac{1}{2})], \\ (s = 1; +1) &= (s_h = \frac{1}{2}; +\frac{1}{2} | s_\ell = \frac{1}{2}; +\frac{1}{2}), \\ (s = 1, 0) &= \frac{1}{\sqrt{2}} [(s_h = \frac{1}{2}; +\frac{1}{2} | s_\ell = \frac{1}{2}; -\frac{1}{2}) + (s_h = \frac{1}{2}; -\frac{1}{2} | s_\ell = \frac{1}{2}; +\frac{1}{2})], \\ (s = 1; -1) &= (s_h = \frac{1}{2}; -\frac{1}{2} | s_\ell = \frac{1}{2}; -\frac{1}{2}). \end{aligned}$$

If one had introduced a phase η to the state (0;0) as $(0;0) = \frac{1}{2\sqrt{2}}(1 + \gamma^0)\eta\gamma_5$, then he/she would have obtained the $(s_h = \frac{1}{2}; +\frac{1}{2} | s_\ell = \frac{1}{2}; -\frac{1}{2})$ state to be $\frac{1}{\sqrt{2}}[(s = 1; 0) + \eta^*(s = 0; 0)]$, while the fact that it is $[\frac{1}{2\sqrt{2}}(1 + \gamma_0)] [\frac{1}{\sqrt{2}}(\gamma^3 + \gamma^5)]$ remains unchanged.

In the heavy mass limit, \vec{I}_h can be rewritten in terms of P_v and P_v^* as

$$\vec{I}_h = v_0 \int d^3r \left\{ P_v \left[-\frac{1}{4} (\xi^\dagger \vec{\tau} \xi + \xi \vec{\tau} \xi^\dagger) \right] P_v^\dagger + P_v^{*i} \left[-\frac{1}{4} (\xi^\dagger \vec{\tau} \xi + \xi \vec{\tau} \xi^\dagger) \right] P_v^{*i\dagger} \right\},$$

and in terms of the H -field as

$$\vec{I}_h = -v_0 \int d^3r \text{Tr} \left\{ H \left[-\frac{1}{4} (\xi^\dagger \vec{\tau} \xi + \xi \vec{\tau} \xi^\dagger) \right] \vec{H} \right\}. \quad (5.13)$$

This implies that, in the rest frame and for the free heavy mesons, the quantum mechanical isospin operators acting on the wavefunction are

$$\vec{I}_h \{h\} = h(\vec{r}, t) (-\frac{1}{2} \vec{\tau}). \quad (5.14)$$

The strange-looking minus sign comes from the fact that the heavy meson fields form *isospin-anti-doublets*. The minus sign is also essential to make the isospin operator defined by Eq. (5.14) obey the correct commutation relations of $SU(2)$ -algebra:

$$[I_h^i, I_h^j] \{h\} = h \left[-\frac{1}{2} \tau^j, -\frac{1}{2} \tau^i \right] = i \varepsilon^{ijk} h \left(-\frac{1}{2} \tau^k \right) = i \varepsilon^{ijk} I_h^k \{h\}.$$

Let the two eigenstates of the isospin operators be $\tilde{\phi}_\pm$ which satisfy

$$\begin{aligned} I_h^3 \{ \tilde{\phi}_\pm \} &= \tilde{\phi}_\pm \left(-\frac{1}{2} \tau^3 \right) = \pm \frac{1}{2} \tilde{\phi}_\pm, \\ I_h^\mp \{ \tilde{\phi}_\pm \} &= \tilde{\phi}_\pm (-\tau^\mp) = \tilde{\phi}_\mp. \end{aligned} \quad (5.15)$$

Explicitly, we have

$$\begin{aligned} \tilde{\phi}_+ &= (0, -1) \sim \text{isospin state of } \bar{d}, \\ \tilde{\phi}_- &= (+1, 0) \sim \text{isospin state of } \bar{u}. \end{aligned} \quad (5.16)$$

First, we combine the orbital angular momentum and the isospin. Let the resulting spherical spinor harmonics be $\mathcal{Y}_{\lambda, \ell, \lambda_3}(\hat{r})$ which are the eigenfunctions of $\vec{\Lambda} (\equiv \vec{L} + \vec{I}_h)$:

$$\mathcal{Y}_{\lambda, \ell, \lambda_3} = \sum_{i_3, m} \left(\ell, m, \frac{1}{2}, i_3 \middle| \lambda, \lambda_3 \right) Y_{\ell m}(\hat{r}) \tilde{\phi}_{i_3}, \quad (5.17)$$

where $(\ell_1, m_1, \ell_2, m_2 | j, m)$ is the Clebsch-Gordan coefficient. Since we are interested in the lowest energy eigenmode of positive parity, we can restrict the angular momentum ℓ to be 1.^{#14} With $\ell=1$, two λ values, $\frac{1}{2}$ and $\frac{3}{2}$, are possible, so we have $\mathcal{Y}_{\frac{1}{2}, 1, \lambda_3}(\hat{r})$ and

^{#14}The heavy mesons have negative intrinsic parity. In general, the differential equations for the radial functions $h_k^i(r)$ have the centrifugal term with a singularity $\ell_{\text{eff}}(\ell_{\text{eff}} + 1)/r^2$ near the origin. It requires for the radial functions to behave as $h_k^i(r) \sim r^{\ell_{\text{eff}}}$ near the origin. Here, ℓ_{eff} is the “effective” angular momentum,¹² which is related to the usual angular momentum ℓ as

$$\ell_{\text{eff}} = \begin{cases} \ell + 1, & \text{if } \lambda = \ell + 1/2, \\ \ell - 1, & \text{if } \lambda = \ell - 1/2. \end{cases}$$

Due to the vector potential from the soliton configuration $\vec{V} (\sim i(\hat{r} \times \vec{\tau})/r, \text{ near the origin})$, the singular structure of $\vec{D}^2 = (\vec{\nabla} - \vec{V})^2$ is altered from $\ell(\ell + 1)/r^2$ of the usual $\vec{\nabla}^2$ to $\ell_{\text{eff}}(\ell_{\text{eff}} + 1)/r^2$. Thus, the states with $\ell_{\text{eff}} = 0$ can have most strongly peaked radial functions and become the lowest eigenstate. Note that $\ell_{\text{eff}} = 0$ can be achieved only when $\ell = 1$.

$\mathcal{Y}_{\frac{3}{2},1,\lambda_3}(\hat{r})$. Explicitly, they are

$$\begin{aligned}\mathcal{Y}_{\frac{1}{2},1,\pm\frac{1}{2}}(\hat{r}) &= +\sqrt{\frac{1}{4\pi}}\tilde{\phi}_{\pm}(\vec{\tau} \cdot \hat{r}), \\ \mathcal{Y}_{\frac{3}{2},1,\pm\frac{3}{2}}(\hat{r}) &= \mp\sqrt{\frac{1}{24\pi}}\tilde{\phi}_{\pm}\tilde{O}_{\pm}(\vec{\tau} \cdot \hat{r}), \\ \mathcal{Y}_{\frac{3}{2},1,\pm\frac{1}{2}}(\hat{r}) &= -\sqrt{\frac{1}{8\pi}}\tilde{\phi}_{\pm}\tilde{O}_0(\vec{\tau} \cdot \hat{r}) = \pm\sqrt{\frac{1}{8\pi}}\tilde{\phi}_{\mp}\tilde{O}_{\pm}(\vec{\tau} \cdot \hat{r}),\end{aligned}\tag{5.18}$$

where $\tilde{O}_i \equiv (\tau^i - 3\hat{r}^i\vec{\tau} \cdot \hat{r})$ with $\tilde{O}_{\pm} = \tilde{O}_1 \pm i\tilde{O}_2$ and $\tilde{O}_0 = \tilde{O}_3$. The explicit factorization of $(\vec{\tau} \cdot \hat{r})$ is for later convenience. Note that the term $[2\vec{\sigma} \cdot \hat{r}\vec{\tau} \cdot \hat{r} - \vec{\sigma} \cdot \vec{\tau}]$ in the equation of motion (5.9) can be simply expressed as $(\vec{\tau} \cdot \hat{r})(\vec{\sigma} \cdot \vec{\tau})(\vec{\tau} \cdot \hat{r})$.

Next, we combine the spin \vec{S} and $\vec{\Lambda}$. We will restrict our consideration to the $k = \frac{1}{2}$ state, which is expected to be the lowest energy state from our experience of the bound-state approach in the Skyrme model.¹² Since we have $s=0, 1$ and $\lambda=\frac{1}{2}, \frac{3}{2}$, we can construct three different grand spin states of $k = \frac{1}{2}$: $\mathcal{K}_{k,k_3}^{(i)}(\hat{r}) (i=1,2,3)$. Explicitly,

$$\begin{aligned}\mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}^{(1)}(\hat{r}) &= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\gamma_5\tilde{\phi}_{\pm}(\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}, \\ \mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}^{(2)}(\hat{r}) &= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\sqrt{\frac{1}{3}}\tilde{\phi}_{\pm}[\vec{\gamma} \cdot \vec{\tau}](\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}, \\ \mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}^{(3)}(\hat{r}) &= [\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\sqrt{\frac{1}{6}}\tilde{\phi}_{\pm}[\vec{\gamma} \cdot \vec{\tau} - 3\vec{\gamma} \cdot \hat{r}\vec{\tau} \cdot \hat{r}](\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}.\end{aligned}\tag{5.19}$$

The eigenstates $\mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}(\hat{r})$ of the equation of motion (5.9) can be expanded in terms of these bases:

$$\mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}(\hat{r}) = \sum_{i=1}^3 c_i \mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}^{(i)}(\hat{r}),\tag{5.20}$$

with the expansion coefficients given by the solution of the secular equation

$$\sum_{j=1}^3 \mathcal{M}_{ij} c_j = -\varepsilon c_i,\tag{5.20a}$$

where the matrix elements \mathcal{M}_{ij} are defined by

$$\begin{aligned}\mathcal{M}_{ij} &= \int d\Omega \text{Tr}\{\mathcal{K}^{(i)}(\hat{r})[\frac{1}{2}gF'(0)(2\vec{\sigma} \cdot \hat{r}\vec{\tau} \cdot \hat{r} - \vec{\sigma} \cdot \vec{\tau})]\bar{\mathcal{K}}^{(j)}(\hat{r})\} \\ &= \int d\Omega \text{Tr}\{\mathcal{K}^{(i)}(\hat{r})(\vec{\tau} \cdot \hat{r})[\frac{1}{2}gF'(0)(\vec{\sigma} \cdot \vec{\tau})](\vec{\tau} \cdot \hat{r})\bar{\mathcal{K}}^{(j)}(\hat{r})\}.\end{aligned}\tag{5.20b}$$

Here again, the minus sign in Eq. (5.20a) is due to the fact that the basis states $\mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}^{(i)}(\hat{r})$ are normalized as (5.8). With the explicit form of $\mathcal{K}_{\frac{1}{2},\pm\frac{1}{2}}^{(i)}(\hat{r})$ given by Eq. (5.19), the matrix elements come out to be

$$\mathcal{M} = -gF'(0) \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -1 & 0 \\ 0 & 0 & +\frac{1}{2} \end{pmatrix},\tag{5.21}$$

Table 5.2 : Eigenvalues and eigenfunctions of $k^\pi = \frac{1}{2}^+$ states.

ε	c_1	c_2	c_3	$\mathcal{K}_{\frac{1}{2}, \pm \frac{1}{2}}(\hat{r})$
$-\frac{3}{2}gF'(0)$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	0	$[\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\frac{1}{2}\tilde{\phi}_\pm[\gamma_5 - \vec{\gamma} \cdot \vec{\tau}](\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}$
$+\frac{1}{2}gF'(0)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$[\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\frac{1}{2\sqrt{3}}\tilde{\phi}_\pm[3\gamma_5 + \vec{\gamma} \cdot \vec{\tau}](\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}$
$+\frac{1}{2}gF'(0)$	0	0	1	$[\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\frac{1}{\sqrt{6}}\tilde{\phi}_\pm[\vec{\gamma} \cdot \vec{\tau} - 3\vec{\gamma} \cdot \hat{r}\vec{\tau} \cdot \hat{r}](\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}$

which is independent of $k_3 (= \frac{1}{2} \text{ or } -\frac{1}{2})$. All the matrix elements between $\mathcal{K}_{\frac{1}{2}, \pm \frac{1}{2}}^{(3)}(\hat{r})$ and $\mathcal{K}_{\frac{1}{2}, \pm \frac{1}{2}}^{(1,2)}(\hat{r})$ vanish. This vanishing can be easily seen from Eq. (5.20b) and the fact that $(\mathcal{K}^{(3)}\vec{\tau} \cdot \hat{r})$ and $(\mathcal{K}^{(1,2)}\vec{\tau} \cdot \hat{r})$ are $\ell=2$ and 0 states, respectively, while $\vec{\sigma} \cdot \vec{\tau}$ cannot have matrix elements between states of different ℓ . The secular equation (5.20a) with these matrix elements yields three eigenstates as listed in Table 5.2. Since $g < 0$ and $F'(0) < 0$ (in the case of the baryon-number-1 soliton solution), we have a heavy-meson-soliton bound state of binding energy $\frac{3}{2}gF'(0)$. The two *unbound* eigenstates with positive eigenenergy $+\frac{1}{2}gF'(0)$ are not consistent with the strongly peaked radial functions. They are improper solutions of Eq. (5.9).

The result is independent of the phase choice for the spin basis ($s = 0; 0$). If we had given a phase η to that state, we would have obtained $\mathcal{M}_{12} = -\frac{\sqrt{3}}{2}\eta gF'(0)$ and $\mathcal{M}_{21} = -\frac{\sqrt{3}}{2}\eta^* gF'(0)$, while the others remain unchanged. But the secular equation must yield the same eigenenergies and the same eigenstates as in Table 5.2 independently of the phase η . In Refs. 30,32, an ansatz $\mathcal{K}_{\frac{1}{2}, \pm \frac{1}{2}}^{NRZ}(\hat{r}) = [\frac{1}{2\sqrt{2}}(1 + \gamma^0)]\frac{1}{2}\tilde{\phi}_\pm[i\eta\gamma_5 - \vec{\gamma} \cdot \hat{r}](\vec{\tau} \cdot \hat{r})\sqrt{\frac{1}{4\pi}}$ was taken as the wavefunction of the lowest energy state. However, it is not the eigenstate of the equation of motion (5.4). It is just one orthonormal basis of the three possible states of $k = \frac{1}{2}$ and $\ell = 1$. The ground state energy of Refs. 30,32, $\frac{3}{4}g_H F'(0)(1 + \frac{1}{2}i(\eta - \eta^*))$ ^{#15}, should be understood as one of the matrix elements similar to Eq. (5.21) and its η -dependence has no physical significance.

Let $h_{\{n\}}$ be the wavefunctions for the eigenmodes classified by the set of quantum number $\{n\} = \{k, k_3, \dots\}$.^{#16} Then, the heavy-meson field $H(x)$ can be expanded in terms of these eigenmodes as

$$H(x) = \sum_{\{n\}} h_{\{n\}}(\vec{r}) e^{-i\varepsilon_{\{n\}}t} a_{\{n\}}, \quad (5.22)$$

with the meson annihilation (creation) operator $a_{\{n\}}$ ($a_{\{n\}}^\dagger$) corresponding to the eigen-

^{#15}The error of factor 3 committed in Ref. 32 is corrected. Note that in Ref. 32 a different sign convention is adopted for the coupling constant; *i.e.*, $g_H = -g$.

^{#16}The ellipsis denotes other quantum numbers, such as parity, radial quantum number etc.

state satisfying the commutation relations

$$[a_{\{n\}}, a_{\{m\}}^\dagger] = \delta_{\{n\}\{m\}}, \quad \text{and} \quad [a_{\{n\}}, a_{\{m\}}] = [a_{\{n\}}^\dagger, a_{\{m\}}^\dagger] = 0.$$

We note that anti-particle creation operators do not figure in this expansion. Focusing on the lowest energy eigenstate (heavy-meson-soliton bound state) for simplicity, we write $H(x)$ explicitly as

$$H(x) = e^{-i\varepsilon_{bs}t}(h_+a_+ + h_-a_-) + \dots \quad (5.23)$$

where the subscript \pm denotes that it is associated with the bound state of $k_3 = \pm\frac{1}{2}$ and $\varepsilon_{bs} = -\frac{3}{2}gF'(0)$. The heavy-meson-soliton bound state of the grand spin $k_3 = \pm\frac{1}{2}$ can be described by the Fock state $a_\pm^\dagger|0\rangle$ with the vacuum state $|0\rangle$ defined by $a_{\{n\}}|0\rangle = 0$ for any $\{n\}$.

Substituting Eq. (5.23) into the heavy-quark spin operator of (4.34) and after normal-ordering, we obtain an interesting result:

$$\begin{aligned} S_h^3 &= \frac{1}{2}a_+^\dagger a_+ - \frac{1}{2}a_-^\dagger a_- + \dots, \\ S_h^+ &= a_+^\dagger a_- + \dots, \\ S_h^- &= a_-^\dagger a_+ + \dots. \end{aligned} \quad (5.24)$$

This shows that the grand spin of the state can be identified as the heavy-quark spin. This is the spin-grand spin transmutation.³² Finally the Hamiltonian (4.35) has a diagonal form of

$$\begin{aligned} H_{\vec{v}=0} &= M_{sol} - g \int d^3r \text{Tr}(H \vec{A} \cdot \vec{\sigma} \bar{H}) \\ &= M_{sol} + \varepsilon_{bs}(a_+^\dagger a_+ + a_-^\dagger a_-) + \dots. \end{aligned} \quad (5.25)$$

5.2. Collective Coordinate Quantization

What we have obtained so far is the heavy-meson-soliton bound state which carries a baryon number and a heavy flavor. Therefore up to this order, baryons containing a heavy quark such as Λ_Q , Σ_Q and Σ_Q^* are degenerate in mass. However, to extract *physical* heavy baryons of correct spin and isospin, we have to go to the next order in $1/N_c$, while remaining in the same order in m_Q ; namely, $O(m_Q^0 N_c^{-1})$. This can be done by quantizing the zero modes associated with the degeneracy under simultaneous $SU(2)$ rotation of the soliton configuration together with the heavy meson fields:

$$\xi_0 \rightarrow C \xi_0 C^\dagger \quad \text{and} \quad H \rightarrow H C^\dagger,$$

with an arbitrary constant $SU(2)$ matrix C and $\xi_0^2 \equiv U_0$. The rotation matrix becomes a dynamical variable when the $SU(2)$ collective variables are endowed time dependence as

$$\xi(\vec{r}, t) = C(t) \xi_0(\vec{r}) C^\dagger(t) \quad \text{and} \quad H(\vec{r}, t) = H_{\text{bf}}(\vec{r}, t) C^\dagger(t), \quad (5.26)$$

and then the quantization is done by elevating the collective variables to the corresponding quantum mechanical operators. In Eq. (5.26), H_{bf} refers to the heavy meson field in the (isospin) co-moving frame, while $H(\vec{r}, t)$ refers to that in the laboratory frame, *i.e.*, the heavy-quark rest frame. Substituting Eq. (5.26) and keeping terms up to $O(m_Q^0 N_c^{-1})$, we obtain the Lagrangian as

$$L = L^1 + L^0 + L^{-1}, \quad (5.27)$$

where L^q denotes the Lagrangian of order $m_Q^0 N_c^q$: *viz.*,

$$L^1 = -M_{\text{sol}}, \quad (5.27a)$$

$$L^0 = \int d^3r \left\{ -i \text{Tr}(\partial_0 H_{\text{bf}} \bar{H}_{\text{bf}}) + g \text{Tr}(H_{\text{bf}} \vec{A} \cdot \vec{\sigma} \bar{H}_{\text{bf}}) \right\}, \quad (5.27b)$$

$$L^{-1} = \frac{1}{2} \mathcal{I} \omega^2 + \int d^3r \left\{ -\frac{1}{4} \text{Tr} \{ H_{\text{bf}} [-\frac{1}{4} (\xi_0^\dagger \vec{\tau} \cdot \vec{\omega} \xi + \xi_0 \vec{\tau} \cdot \vec{\omega} \xi_0^\dagger)] \bar{H}_{\text{bf}} \} \right\}. \quad (5.27c)$$

The “angular velocity” $\vec{\omega}$ of the collective rotation is defined by

$$C^\dagger \partial_0 C \equiv \frac{1}{2} i \vec{\tau} \cdot \vec{\omega}, \quad (5.27d)$$

and \mathcal{I} is the moment of inertia of the rotating soliton

$$\mathcal{I} = \frac{8\pi}{3} f_\pi^2 \int_0^\infty r^2 dr (\sin^2 F + \dots), \quad (5.27e)$$

where the ellipsis stands for contributions from higher-derivative terms (such as the quartic term etc.). Note that V_0 does not vanish for the “rotating” soliton, so once again the covariant derivative D_0 plays a non-trivial role as in Eq. (5.12b).

Given the Lagrangian (5.27) that describes dynamics up to order $O(m_Q^0 N_c^{-1})$, one has the equation of motion consistent to that order:

$$i \partial_0 H_{\text{bf}} = H_{\text{bf}} \left\{ g \vec{A} \cdot \vec{\sigma} - \frac{1}{4} (\xi_0^\dagger \vec{\tau} \cdot \vec{\omega} \xi + \xi_0 \vec{\tau} \cdot \vec{\omega} \xi_0^\dagger) \right\}. \quad (5.27)$$

Note that the last “Coriolis” term in the equation of motion couples the fast and slow degrees of freedom.⁷⁴ Although the heavy mesons are infinitely heavy, their angular momentum and the isospin are associated with the light constituents. Thus, we may take those light degrees of freedom of the heavy meson fields as “fast” variables and the collective rotation as “slow” variables. Note further that the scale of the eigenenergies $|\varepsilon_n|$ of the heavy mesons is much greater than that of the rotational velocity; $|\varepsilon_n| \gg |\omega|$.

A generally accepted procedure of handling these different scales is as follows. We first solve the equation of motion for fast degrees of freedom with slow degrees of freedom “frozen.” In this way, we get “snap-shot” pictures of the fast motion. Next we solve the equation of motion for slow degrees of freedom taking into account the “relic” of the fast motion that has been “integrated out,” in a manner completely analogous to the incorporation of Berry phases.⁷⁵ In Ref. 32, it is shown that the “Coriolis effect” on the heavy-meson-soliton system does not induce³⁴ any non-trivial Berry phase as far as the $k^P = \frac{1}{2}^+$ multiplets are concerned. It is also analogous to the “strong-coupling

limit” of the particle-rotor model⁷⁶ in nuclear physics, where the coupling between the rotating “core” and the particle is much stronger than the perturbation of the single-particle motion by a Coriolis interaction. Here, the roles of the particle and the rotor are played by the bound heavy-mesons and the rotating soliton configuration. Thus, we may make the assumption that the bound heavy mesons rotate together with the soliton core in the *unchanged eigenmodes*. It enables us to expand the $H_{\text{bf}}(x)$ in terms of the classical eigenmodes obtained in Sec. 5.1 as

$$H_{\text{bf}}(x) = \sum_{\{n\}} h_{\{n\}}(\vec{r}) e^{-i\varepsilon_{\{n\}}t} a_{\{n\}} = e^{-i\varepsilon_{bs}t} (h_+ a_+ + h_- a_-) + \cdots, \quad (5.28)$$

and to describe the heavy-meson soliton bound state by the Fock state $|\pm \frac{1}{2}\rangle_{bs} \equiv a_{\pm}^{\dagger}|0\rangle$.

Taking the Legendre transform of the Lagrangian (5.27)^{#17} we obtain the Hamiltonian as

$$\begin{aligned} H_{\vec{v}=0} &= \int d^3r \left\{ \frac{\delta \mathcal{L}}{\delta(\dot{H}_{\text{bf},ab})} \dot{H}_{\text{bf},ab} \right\} + \frac{\delta L}{\delta \omega_i} \omega_i - L \\ &= M_{\text{sol}} - g \int d^3r \text{Tr}(H_{\text{bf}} \vec{A} \cdot \vec{\sigma} \bar{H}_{\text{bf}}) + \frac{1}{2\mathcal{I}} [\vec{J}_R - \vec{\Phi}(\infty)]^2, \end{aligned} \quad (5.29)$$

where \vec{J}_R is the canonical momenta conjugate to the collective variables $C(t)$:

$$J_R^i \equiv \frac{\delta L_0^{\text{rot}}}{\delta \omega^i} = \mathcal{I} \omega^i + \Phi^i(\infty), \quad (5.29a)$$

with $\vec{\Phi}(\infty)$ defined by

$$\vec{\Phi}(\infty) \equiv - \int d^3r \text{Tr} \left\{ H_{\text{bf}\frac{1}{4}} (\xi_0^{\dagger} \vec{\tau} \xi + \xi_0 \vec{\tau} \xi_0^{\dagger}) [\bar{H}_{\text{bf}}] \right\}, \quad (5.29b)$$

whose expectation value with respect to the state $|\pm \frac{1}{2}\rangle_{bs}$ is the Berry phase *in the heavy-quark symmetry limit* associated with the collective rotation. It can be easily shown that this Berry phase vanishes identically; *viz.*,

$$\begin{aligned} \langle \vec{\Phi}(\infty) \rangle &\equiv {}_{bs} \langle \pm \frac{1}{2} | - \int d^3r \text{Tr} (H_{\text{bf}\frac{1}{4}} (\xi_0^{\dagger} \vec{\tau} \xi + \xi_0 \vec{\tau} \xi_0^{\dagger}) [\bar{H}_{\text{bf}}]) | \pm \frac{1}{2} \rangle_{bs} \\ &= -\frac{1}{2} \int d\Omega \text{Tr} (h_{\pm} (\vec{\tau} \cdot \hat{r}) \vec{\tau} (\vec{\tau} \cdot \hat{r}) \bar{h}_{\pm}) = 0. \end{aligned}$$

Note that $\vec{\Phi}(\infty)$ is the isospin operator of heavy mesons (in the body fixed frame) modulo the sign.

With the collective variable introduced as in Eq. (5.26), the isospin of the fields $U(x)$ and $H(x)$ is entirely shifted to $C(t)$. To see this, consider the isospin rotation

$$U \rightarrow \mathcal{A} U \mathcal{A}^{\dagger}, \quad H \rightarrow H \mathcal{A}^{\dagger},$$

^{#17}One can get the same Hamiltonian by substituting Eq. (5.26) into the Hamiltonian (4.35) and keeping terms up to $O(m_Q^0 N_C^{-1})$.

with $\mathcal{A} \in SU(2)_V$, under which the collective variables and fields in body-fixed frame transform as

$$C(t) \rightarrow \mathcal{A}C(t), \quad H_{\text{bf}}(x) \rightarrow H_{\text{bf}}(x). \quad (5.30)$$

The H -field is isospin-blind in the (isospin) co-moving frame. The conventional Noether construction gives the isospin of the system,

$$I^a = \frac{1}{2} \text{Tr}(\tau^a C \tau^b C^\dagger) [\mathcal{I}\omega^b + \Phi^b(\infty)] = D^{ab}(C) J_R^b, \quad (5.31)$$

where $D^{ab}(C)$ is the adjoint representation of the $SU(2)$ associated with the collective variables $C(t)$. One may also obtain the same isospin by substituting Eq. (5.26) into Eq. (5.12) and keeping terms up to $O(m_Q^0 N_c^{-1})$:

$$\begin{aligned} I^i &= \int d^3r \left\{ \left(i\frac{1}{2} f_\pi^2 \text{Tr}[\frac{1}{2} \tau^i (U^\dagger \partial_0 U + U \partial_0 U^\dagger)] + \dots \right) + \text{Tr}[H(\frac{1}{2} \tau^i) \bar{H}] \right\} \\ &= \frac{1}{2} \text{Tr}(\tau^i C \tau^j C^\dagger) [\mathcal{I}\omega^j + \Phi^j(\infty)]. \end{aligned}$$

Under the spatial rotation, with the help of the K -symmetry, the fields transform as

$$\begin{aligned} U(\vec{r}, t) &\rightarrow e^{i\vec{\alpha} \cdot \vec{L}} U(\vec{r}', t) = C(t) \mathcal{B}^\dagger U_0(\vec{r}) \mathcal{B} C^\dagger(t), \\ H(\vec{r}, t) &\rightarrow e^{i\vec{\alpha} \cdot \vec{\sigma}/2} H(\vec{r}', t) e^{-i\vec{\alpha} \cdot \vec{\sigma}/2} = [e^{i\vec{\alpha} \cdot \vec{\sigma}/2} H_{\text{bf}}(\vec{r}', t) e^{-i\vec{\alpha} \cdot (\vec{\sigma} + \vec{\tau})/2}] \mathcal{B}^\dagger C(t), \end{aligned}$$

with $\vec{r}' = \exp(i\vec{\alpha} \cdot \vec{L}) \vec{r}$ and $\mathcal{B} = \exp(i\vec{\alpha} \cdot \vec{\tau}/2) \in SU(2)$. This means that the spatial rotation acts on the collective variables and H -fields in the body fixed frame as

$$\begin{aligned} C(t) &\rightarrow C(t) \mathcal{B}^\dagger, \\ H_{\text{bf}}(x) &\rightarrow e^{i\vec{\alpha} \cdot \vec{L}} e^{i\vec{\alpha} \cdot \vec{\sigma}/2} H_{\text{bf}}(x) e^{-i\vec{\alpha} \cdot (\vec{\sigma} + \vec{\tau})/2}. \end{aligned} \quad (5.32)$$

Therefore, we see that the spin of the $H_{\text{bf}}(x)$ is the grand spin, *i.e.*, the spin–grand-spin transmutation. The $H_{\text{bf}}(x)$ becomes isospin-blind; that is, the isospin of the H -field is transmuted into the part of the spin in the isospin co-moving frame. Applying the Noether theorem to the Lagrangian (5.27), we obtain the spin of the system explicitly as

$$\vec{J} = \vec{J}_R + \vec{K}_{\text{bf}}, \quad (5.33)$$

with the grand spin of the heavy meson fields (in the body-fixed coordinate system)

$$\vec{K}_{\text{bf}} = \int d^3r \text{Tr} \left\{ \left\{ \vec{L} H_{\text{bf}} + \left[\frac{1}{2} \vec{\sigma}, H_{\text{bf}} \right] + H_{\text{bf}} \left(-\frac{1}{2} \vec{\tau} \right) \right\} \bar{H}_{\text{bf}} \right\}.$$

We should point out that the heavy-quark spin symmetry of the Lagrangian under the transformation

$$H(x) \rightarrow e^{i\vec{\alpha} \cdot \vec{\sigma}/2} H(x) = [e^{i\vec{\alpha} \cdot \vec{\sigma}/2} H_{\text{bf}}(x)] C(t), \quad (5.34)$$

has nothing to do with the collective rotation. The heavy-quark spin operator remains unchanged in the isospin co-moving frame:

$$\vec{S}_Q = - \int d^3r \text{Tr}(\frac{1}{2}\vec{\sigma}H\bar{H}) = - \int d^3r \text{Tr}(\frac{1}{2}\vec{\sigma}H_{\text{bf}}\bar{H}_{\text{bf}}). \quad (5.35)$$

Upon canonical quantization, the collective variables become the quantum mechanical operators; the isospin (I), the spin (J) and the spin of the rotor (J_R) discussed so far become the corresponding operators \tilde{I}^i , \tilde{J}^i and \tilde{J}_R , respectively. We distinguish those operators associated with the collective coordinate quantization by a tilde. The eigenfunctions of the rotor-spin operator \tilde{J}_R^i are the Wigner D -functions

$$\sqrt{2I+1}D_{MK}^{(I)}(C), \quad (5.36)$$

with $M, K = -I, -I+1, \dots, I$ and

$$\tilde{J}_R^i \tilde{J}_R^i D_{MK}^{(I)}(C) = I(I+1)D_{MK}^{(I)}(C), \quad (5.36a)$$

$$\tilde{J}_R^3 D_{MK}^{(I)}(C) = -KD_{MK}^{(I)}(C), \quad (5.36b)$$

$$D^{3i} \tilde{J}_R^i D_{MK}^{(I)}(C) = MD_{MK}^{(I)}(C). \quad (5.36c)$$

Now, in terms of these eigenfunctions and the bound-heavy-meson state $|\pm \frac{1}{2}\rangle_{bs}$, the heavy baryon state of isospin i_3 and spin s_3 containing a heavy quark can be constructed as

$$\begin{aligned} |\Sigma_Q; i_3, s_3\rangle &= \sqrt{3} \sum_m (1, s_3 - m, \frac{1}{2}, m | \frac{1}{2}, s_3) D_{1, -s_3+m}^{(1)}(C) |m\rangle_{bs}, \\ |\Sigma_Q^*; i_3, s_3\rangle &= \sqrt{3} \sum_m (1, s_3 - m, \frac{1}{2}, m | \frac{3}{2}, s_3) D_{1, -s_3+m}^{(1)}(C) |m\rangle_{bs}, \\ |\Lambda_Q; 0, s_3\rangle &= D_{0,0}^{(0)}(C) |s_3\rangle_{bs}, \end{aligned} \quad (5.37)$$

where $(j_1, m_1, j_2, m_2 | j, m)$ is the usual Clebsch-Gordan coefficient. To give correct quantum numbers to the baryons, we have quantized the rotor as a boson.⁷⁷ They are the eigenstates of the collective Hamiltonian (5.29) with the eigenenergies

$$\begin{aligned} E_{\Sigma_Q} &= E_{\Sigma_Q^*} = M_{sol} + \varepsilon_{bs} + \frac{11}{8\mathcal{I}}, \\ E_{\Lambda_Q} &= M_{sol} + \varepsilon_{bs} + \frac{3}{8\mathcal{I}}. \end{aligned} \quad (5.38)$$

Here, we have *exactly* evaluated the expectation value of the operator $\vec{\Phi}^2(\infty)$ with respect to the Fock state $|m\rangle_{bs}$ ($m = \pm \frac{1}{2}$) and obtained⁷⁸

$${}_{bs}\langle m | \vec{\Phi}^2(\infty) | m \rangle_{bs} = \frac{3}{4}, \quad (5.39)$$

instead of taking into account only the bound state contribution

$${}_{bs}\langle m|\vec{\Phi}^2(\infty)|m\rangle_{bs} = \sum {}_{bs}\langle m|\vec{\Phi}|n\rangle \cdot \langle n|\vec{\Phi}|m\rangle_{bs} \approx |{}_{bs}\langle m|\vec{\Phi}(\infty)|m\rangle_{bs}|^2 = 0. \quad (5.40)$$

One can show Eq. (5.39) by carrying out the summation in Eq. (5.40) over the complete set of energy eigenstates.

With vanishing Berry phase, the Hamiltonian depends only on the rotor-spin so that Σ_Q and Σ_Q^* become degenerate as expected from the heavy-quark symmetry. In order to compare the results with experimental heavy baryon masses, we have to add the heavy meson masses subtracted so far from the eigenenergies. The mass formulas to be compared with data are

$$\begin{aligned} m_{\Sigma_Q} &= m_{\Sigma_Q^*} = M_{sol} + \overline{m}_P - \frac{3}{2}gF'(0) + \frac{11}{8\mathcal{I}}, \\ m_{\Lambda_Q} &= M_{sol} + \overline{m}_P - \frac{3}{2}gF'(0) + \frac{3}{8\mathcal{I}}, \end{aligned} \quad (5.41)$$

where \overline{m}_P is the weighted average of the heavy meson multiplets; $\overline{m}_P = \frac{1}{4}(3m_{P^*} + m_P)$. In the case of $Q = c$, we have $\overline{m}_P = 1975$ MeV. The $SU(2)$ quantities M_{sol} and \mathcal{I} are obtained from the nucleon and Δ masses^{72#18}:

$$M_{sol} = 866 \text{ MeV}, \quad \text{and} \quad 1/\mathcal{I} = 195 \text{ MeV}. \quad (5.42)$$

Finally, the unknown value of $gF'(0)$ is adjusted to fit the observed value of the Λ_c mass,

$$m_{\Lambda_c} = 2285 \text{ MeV} = M_{sol} + \overline{m}_P - \frac{3}{2}gF'(0),$$

which implies that

$$gF'(0) = 419 \text{ MeV}. \quad (5.43)$$

This set of parameters leads to a prediction on the Λ_b mass and the average mass of the Σ_Q - Σ_Q^* multiplets, $\overline{m}_{\Sigma_Q} [\equiv \frac{1}{3}(2m_{\Sigma_Q^*} + m_{\Sigma_Q})]$,

$$\begin{aligned} m_{\Lambda_b} &= M_{sol} + \overline{m}_B - \frac{3}{2}gF'(0) + 3/8\mathcal{I} = 5623 \text{ MeV}, \\ \overline{m}_{\Sigma_c} &= M_{sol} + \overline{m}_D - \frac{3}{2}gF'(0) + 11/8\mathcal{I} = 2480 \text{ MeV}. \end{aligned} \quad (5.44)$$

These are comparable with the experimental masses of Λ_b (5641 MeV) and Σ_c (2453 MeV)¹⁰ and Σ_c^* (2530 MeV).¹¹ Furthermore, with the Skyrme Lagrangian (with the quartic term for stabilization), the wavefunction has a slope $F'(0) \sim -2ef_\pi \approx -690$ MeV^{#19} near the origin, which implies $g \sim -0.61$. This is also consistent with the experimental limit $g^2 < 0.5$ and with the nonrelativistic quark model prediction $g = -\frac{3}{4}$.

^{#18}As discussed elsewhere,⁵⁶ we could do better by calculating the $O(N_c^0)$ Casimir energy which is of the order of $-1/2$ GeV and fitting the N and Δ spectrum to obtain the parameters of the $SU(2)$ sector. Unfortunately the Casimir energy calculation is not yet sufficiently accurate enough to be quantitatively useful at present. We believe that the procedure used here is not really satisfactory and could certainly be improved upon when the Casimir calculation is put under control.

^{#19}In order to yield $M_{sol} = 866$ MeV and $1/\mathcal{I} = 195$ MeV, the Skyrme parameter e and the meson f_π are adjusted to be 5.45 and 63 MeV, respectively. See Ref. 72.

5.3. Alternative Approach

Up to now, we have discussed how one can obtain the heavy baryon states containing a heavy quark, Σ_Q , Σ_Q^* and Λ_Q , as heavy-meson-soliton bound states treated in the standard way: a heavy-meson-soliton bound state is first found and then quantized by rotating the *whole* system in the collective coordinate quantization scheme. This amounts to proceeding systematically in a decreasing order in N_c ; that is, in the first step, only terms up to N_c^0 order are considered and in the next step, terms of order $1/N_c$ order and so on. In this way of proceeding, the heavy mesons first lose their quantum numbers (such as the spin and isospin), with only the grand spin preserved. The good quantum numbers are recovered when the whole system is quantized properly.

An alternative approach adopted in Ref. 28 is more natural in the “top-down” approach. In this approach, the soliton is first quantized to produce the light baryon states such as nucleons and Δ ’s with correct quantum numbers. Then, the heavy mesons with explicit spin and isospin are coupled to the light baryons to form heavy baryons as a bound state. Compared with the traditional one which is a “body-fixed” approach, this approach may be interpreted as a “laboratory-frame” approach.

To start with, we redefine the meson fields so that the Lagrangian density is expressed in terms of U instead of ξ which has a coordinate singularity. The new heavy meson fields P'_v and $P'_{v\mu}$ defined by

$$H' = \frac{1 + \psi'}{2}(\gamma_5 P'_v - \gamma^\mu P'_{v\mu}) = H\xi, \quad (5.45)$$

transform under chiral $SU(2)_L \times SU(2)_R$ (2.17) as

$$H' \rightarrow H' R^\dagger. \quad (5.46)$$

The Lagrangian density (4.33) now reads

$$\mathcal{L} = \mathcal{L}_M - iv^\mu \text{Tr}(\partial_\mu H' \bar{H}') + \frac{i}{2} v^\mu \text{Tr}(H' U^\dagger \partial_\mu U \bar{H}') - \frac{i}{2} g \text{Tr}(H' \gamma_5 U^\dagger \partial_\mu U \gamma^\mu \bar{H}'). \quad (5.47)$$

As was discussed in Sec. 2, the parity transformation for the primed heavy meson fields is a little more complicated:

$$H'(t, \vec{r}) \rightarrow \gamma^0 H'(t, -\vec{r}) \gamma^0 U^\dagger(t, -\vec{r}), \quad (5.48)$$

compared with that for the unprimed fields

$$H(t, \vec{r}) \rightarrow \gamma^0 H(t, -\vec{r}) \gamma^0.$$

Note that in the background Goldstone boson field configuration of a soliton located at the same spatial point as the heavy meson, the factor of U^\dagger becomes -1 , whereas $U^\dagger = 1$ for a meson infinitely far from the soliton. This relative minus sign is the source of the parity flip that gives positive parity heavy-meson-soliton bound states.

The (approximate) soliton solution of the $SU(2)_L \times SU(2)_R$ chiral Lagrangian \mathcal{L}_M

$$U(\vec{r}, t) = C(t) U_0(\vec{r}) C^\dagger(t),$$

is substituted into the Lagrangian (4.33). In the rest frame, up to order $1/N_c$, the Lagrangian reads

$$L_0 = -M_{sol} + \frac{1}{2}\mathcal{I}\omega^2 - \int d^3r \left\{ i \text{Tr}(\partial_0 H' \bar{H}') + \frac{i}{2} \text{Tr}(H' U^\dagger \partial_0 U \bar{H}') + \frac{ig}{2} \text{Tr}(H' \gamma_5 \vec{\gamma} \cdot [U^\dagger \vec{\nabla} U] \bar{H}') \right\}. \quad (5.49)$$

The $1/N_c$ terms with time derivative on U are neglected as we are assuming the large- N_c limit at which the soliton is very heavy.

First collective quantization of the soliton leads to the light baryon states $|s, s_3; i, i_3\rangle_{lb}$ with the wavefunctions given by the Wigner D -functions $D_{MK}^{(I)}(C)$:

$$\begin{aligned} \phi_{s=\frac{1}{2}, s_3; i=\frac{1}{2}, i_3}^N(C) &= \sqrt{2} D_{i_3, -s_3}^{(\frac{1}{2})}(C) \quad \text{for nucleons,} \\ \phi_{s=\frac{3}{2}, s_3; i=\frac{3}{2}, i_3}^\Delta(C) &= \sqrt{4} D_{i_3, -s_3}^{(\frac{3}{2})}(C) \quad \text{for deltas.} \end{aligned} \quad (5.50)$$

In the large- N_c limit, these states are degenerate. Next, the interaction Hamiltonian

$$H_I = -\frac{ig}{2} \int d^3r \text{Tr}(H' \gamma_5 \vec{\gamma} \cdot [U^\dagger \vec{\nabla} U] \bar{H}'), \quad (5.51)$$

determines the potential energy of a configuration with a baryon (soliton) sitting at the origin and the heavy mesons at position \vec{r} . Assuming that in attractive channels the potential energy is minimized at $\vec{r} = \vec{0}$ where the heavy meson and the baryon soliton coincide, we can reduce the problem of determining the bound-state spectrum to finding the eigenvalues of the potential operator at the origin. Near the origin, we have

$$\begin{aligned} U^\dagger \vec{\nabla} U &= iC(t) \left\{ \vec{\tau} \cdot \hat{r} \hat{r}(F' - \frac{\sin 2F}{2r}) + \vec{\tau} \frac{\sin 2F}{2r} + \vec{\tau} \times \hat{r} \frac{\sin^2 F}{r} \right\} C^\dagger(t) \\ &= iF'(0)C(t) \vec{\tau} C^\dagger(t). \end{aligned}$$

The interaction energy is therefore

$$\begin{aligned} V_I(0) &= \frac{1}{2} g F'(0) \int d^3r \text{Tr}(H' \gamma_5 \vec{\gamma} \cdot [C(t) \vec{\tau} C^\dagger(t)] \bar{H}') \\ &= 2g F'(0) S_\ell^a I_h^b D^{ba}(C), \end{aligned} \quad (5.52)$$

where S_ℓ and I_h are the spin operator of the light degrees of freedom and the isospin operator acting on the heavy meson states, respectively, and we have used that

$$C(t) \tau^a C^\dagger(t) = \tau^b D^{ba}(C), \quad (5.52a)$$

and that

$$S_\ell^a I_h^b = - \int d^3r \text{Tr}(H' \{ -\frac{1}{2} \sigma^a \} \{ -\frac{1}{2} \tau^b \} \bar{H}'). \quad (5.52b)$$

(See the discussions in Sec. 4.2 and 5.1 on the spin and isospin operators.)

Note that the potential operator is invariant under rotation by the total angular momentum operator of the light degrees of freedom (both soliton and light anti-quarks in heavy meson combined) $J_\ell^a (\equiv S_\ell^a + \tilde{J}_R^a)$ and under rotation by the total isospin operator $I^a \equiv I_h^a + \tilde{I}_R^a$. Furthermore, it is completely independent of the heavy quark spin as required by the heavy-quark spin symmetry. Thus, the eigenstates of the potential operator can be classified by the corresponding quantum numbers $j_\ell, j_{\ell 3}, i, i_3$ and s_{h3} ; viz., $|j_\ell, j_{\ell 3}; i, i_3\rangle |s_h = \frac{1}{2}, s_{h3}\rangle$. Let the eigenstates of \vec{J}_ℓ and \vec{I} be denoted by $|j_\ell, j_{\ell 3}; i, i_3\rangle$, constructed by combining the solitonic light baryon states $|s_R, s_{R3}; i_R, i_{R3}\rangle$ and the light degrees of freedom in the heavy meson states $|s_\ell, s_{\ell 3}; i_h, i_{h3}\rangle$:

$$|j_\ell, j_{\ell 3}; i, i_3\rangle = \sum_{s_{R3}, i_{R3}} (s_R, s_{R3}, s_\ell, s_{\ell 3} | j_\ell, j_{\ell 3}) (i_R, i_{R3}, i_h, i_{h3} | i, i_3) \cdot |s_R, s_{R3}; i_R, i_{R3}\rangle |s_\ell, s_{\ell 3}; i_h, i_{h3}\rangle, \quad (5.53)$$

with the appropriate Clebsch-Gordan coefficients. Since we have the light baryon states of $i = s = 1/2, 3/2, \dots$, and $i_h = s_\ell = 1/2$ for the light degrees of freedom of the heavy mesons, we can construct the states with $j_\ell, i = 0, 1, \dots$. Hereafter, unless necessary, we will not specify the third components explicitly and simply denote the basis states as $|j_\ell, i\rangle$, as the eigenstates are degenerate in $j_{\ell 3}$ and i_3 . Finally the eigenstates of the potential operator are found by diagonalizing the potential matrix calculated with the states $|j_\ell, i\rangle$ as basis states.

When the basis is truncated with only nucleon-heavy meson products, it is straightforward to show that

$$V_I(0) = -\frac{2}{3}gF'(0)S_\ell^a I_h^b \tilde{J}_R^a \tilde{I}_R^b = -\frac{2}{3}gF'(0)(J_\ell^2 - 3/2)(I^2 - 3/2). \quad (5.54)$$

This implies that the basis states $|j_\ell, i\rangle$ themselves are the approximate eigenstates with eigenenergies $-\frac{2}{3}gF'(0)[j_\ell(j_\ell + 1) - 3/2][i(i + 1) - 3/2]$.^{#20} In getting this result, we have used the fact that $D^{ab}(C)$ can be written in terms of the D -functions, namely,

$$D^{3,3}(C) = D_{0,0}^{(1)}(C), \quad \mp \frac{1}{\sqrt{2}}[D^{1,3}(C) \pm iD^{2,3}(C)] = D_{\pm 1,0}^{(1)}(C), \quad (5.55)$$

and that

$$\int dC D_{m'_1 m_1}^{(j_3)*}(C) D_{m'_2 m_2}^{(j_2)}(C) D_{m'_1 m_1}^{(j_1)}(C) = \frac{1}{2j_3 + 1} (j_1 m'_1 j_2 m'_2 | j_3 m'_3) (j_1 m_1 j_2 m_2 | j_3 m_3), \quad (5.56)$$

with the Clebsch-Gordan coefficients $(\dots | \dots)$. Specifically, the expectation values with respect to the nucleon states are obtained as

$$\{\frac{1}{2}, s'_3; \frac{1}{2}, i'_3 | D^{ab}(C) | \frac{1}{2}, s_3; \frac{1}{2}, i_3\} = -\frac{4}{3} \{\frac{1}{2}, s'_3; \frac{1}{2}, i'_3 | \tilde{J}_R^a \tilde{I}_R^b | \frac{1}{2}, s_3; \frac{1}{2}, i_3\}. \quad (5.57)$$

^{#20}More precisely, this should be interpreted as matrix elements of the potential operator.

Table 5.3: Eigenstates and Eigenenergies of $V_I(0)$.

States	Energies neglecting the Δ	Energies including the Δ
$ 0, \frac{1}{2}, 0\rangle$	$-\frac{3}{2}gF'(0)$	$-\frac{3}{2}gF'(0)$
$ 1, \frac{1}{2}, 1\rangle, 1, \frac{3}{2}, 1\rangle$	$-\frac{1}{6}gF'(0)$	$-\frac{3}{2}gF'(0)$
$ 1, \frac{1}{2}, 0\rangle$	$+\frac{1}{2}gF'(0)$	$+\frac{1}{2}gF'(0)$
$ 0, \frac{1}{2}, 1\rangle, 0, \frac{3}{2}, 1\rangle$	$+\frac{1}{2}gF'(0)$	$+\frac{1}{2}gF'(0)$

The eigenstate of $V_I(0)$ with definite isospin and spin can be obtained by combining the heavy quark spin to the total angular momentum for the light degrees of freedom. Let these eigenstates be denoted by $|i, j, j_\ell\rangle$, where i is the total isospin, j the total spin ($\vec{J} = \vec{J}_\ell + \vec{S}_h$) and j_ℓ the angular momentum of the light degrees of freedom. Given in Table 5.3 are the eigenstates and eigenvalues of $V_I(0)$ in the truncated basis. Only the $|0, \frac{1}{2}, 0\rangle$, $|1, \frac{1}{2}, 1\rangle$ and $|1, \frac{3}{2}, 1\rangle$ states are bound. For the case of $h = c$, these states have the right quantum numbers i, j, j_ℓ and the parity^{#21} to be Λ_c , Σ_c and Σ_c^* , respectively.

In the large- N_c limit, the N and Δ are degenerate and the space of the basis states should be enlarged to include products of Δ -baryons with the heavy-mesons. There appear two states in the $j_\ell = i = 1$ channel, obtained by combining the light degrees of freedom of the heavy mesons with either the N ($s_R = i_R = 1/2$) or the Δ ($s_R = i_R = 3/2$) states. These states will be distinguished by explicitly specifying the light baryon states as $|1, 1; N\rangle$ and $|1, 1; \Delta\rangle$, where the first label refers to the isospin and the second to the spin of the light degrees of freedom. One can evaluate the interaction Hamiltonian Eq. (5.54) in the $|1, 1; N\rangle$ and $|1, 1; \Delta\rangle$ basis as

$$V_I(0) = -\frac{gF'(0)}{6} \begin{pmatrix} 1 & 4\sqrt{2} \\ 4\sqrt{2} & 5 \end{pmatrix}. \quad (5.58)$$

Diagonalizing it, we can obtain two $|1, 1\rangle$ eigenstates of $V_I(0)$ as

$$\begin{aligned} |1, 1\rangle_- &= \sqrt{\frac{1}{3}} |1, 1; N\rangle + \sqrt{\frac{2}{3}} |1, 1; \Delta\rangle, \\ |1, 1\rangle_+ &= \sqrt{\frac{2}{3}} |1, 1; N\rangle - \sqrt{\frac{1}{3}} |1, 1; \Delta\rangle, \end{aligned} \quad (5.59)$$

^{#21}The spatial wave functions of the eigenstates of Eq. (5.54) carry zero orbital angular momentum. The parity of the meson-soliton bound state is thus even, because the primed heavy-meson fields are odd under parity at infinity where they are free, and are even under parity at the origin, as noted below Eq. (5.48). The unprimed heavy-meson fields have a simple transformation law under parity, and do not have a relative minus sign between the parity at infinity and parity at the origin. However, in the ξ basis, the wavefunction of the bound state contains a factor of the form $\vec{\tau} \cdot \hat{r}$ near the origin and it is the state of $\ell = 1$. (See Sec. 5.1)

with eigenenergies $-3gF'(0)/2$ and $gF'(0)/2$, respectively. In the large N_c limit, we see that the states $|0, 0\rangle\rangle$ and $|1, 1\rangle\rangle_0$ are degenerate. Again, the former, when combined with the heavy quark spin, is the spin-1/2 Λ_Q baryon, and the state $|1, 1\rangle\rangle_0$ when combined with the heavy quark spin is the degenerate multiplet of Σ_Q and the spin 3/2 Σ_Q^* : *viz.*,

$$\begin{aligned} |\Sigma_Q^*, \tfrac{3}{2}\rangle &= |1\ 1\ 1\rangle\rangle_- |\uparrow\rangle_Q, \\ |\Sigma_Q, \tfrac{1}{2}\rangle &= \sqrt{\tfrac{2}{3}}|1\ 1\ 1\rangle\rangle_- |\downarrow\rangle - \sqrt{\tfrac{1}{3}}|1\ 1\ 0\rangle\rangle_0 |\uparrow\rangle, \\ |\Lambda_Q, \tfrac{1}{2}\rangle &= |0\ 0\ 0\rangle\rangle |\uparrow\rangle. \end{aligned} \quad (5.60)$$

Here, the state $|\rangle_Q$ denotes the heavy quark spin state and $|i\ j_\ell\ m\rangle\rangle$ represents the state of the light degrees of freedom $|i\ j_\ell\rangle\rangle$ with $j_{\ell 3}=m$.

In the large N_c -limit, Λ_Q and Σ_Q are also degenerate. The degeneracy is lifted when the terms of order $1/N_c$ so far neglected are taken into account. There are two $1/N_c$ -order terms in Eq. (5.49). The first is the rotational kinetic energy of the soliton that lifts the N - Δ degeneracy. This term has a coefficient of order N_c (the moment of inertia) in the Lagrangian, and has two time derivatives each of which brings a factor of $1/N_c$ suppression so that it produces an energy splitting of order $1/N_c$. It contributes additional masses, $3/8\mathcal{I}$ and $15/8\mathcal{I}$, to the nucleon and delta masses, respectively, and thus it yields the N - Δ mass splitting of $3/2\mathcal{I}$. The second is the term with $V_0 = \frac{1}{2}U^\dagger\partial_0 U$, having a coefficient of order N_c^0 and one time derivative. For the $SU(2)$ soliton, however, $U^\dagger\partial_0 U$ vanishes at the origin where the heavy meson is bound. When the additional masses of the light baryon states are included, the interaction energy of the Λ_h state, $|0, \frac{1}{2}, 0\rangle$, is modified to $-3gF'(0)/2 + 3/8\mathcal{I}$. Thus the potential matrix in the $|1, 1; N\rangle$ and $|1, 1; \Delta\rangle$ channel is of the form^{28,29}

$$V_I(0) = -\frac{gF'(0)}{6} \begin{pmatrix} 1 & 4\sqrt{2} \\ 4\sqrt{2} & 5 \end{pmatrix} + \frac{3}{8\mathcal{I}} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}. \quad (5.61)$$

To first order in $1/\mathcal{I}$, the eigenvalues come out as

$$E_- = -\frac{3gF'(0)}{2} + \frac{11}{8\mathcal{I}}, \quad E_+ = +\frac{gF'(0)}{2} + \frac{7}{8\mathcal{I}}. \quad (5.62)$$

The eigenenergy E_- leads to the same heavy baryon masses as Eq. (5.41). The corresponding eigenstate is

$$|1, 1\rangle\rangle_\epsilon = a|1, 1; N\rangle + b|1, 1; \Delta\rangle = |1, 1\rangle\rangle_- + \epsilon|1, 1\rangle\rangle_+, \quad (5.63)$$

where

$$a = \sqrt{\tfrac{1}{3}} + \sqrt{\tfrac{2}{3}}\epsilon, \quad b = \sqrt{\tfrac{2}{3}} - \sqrt{\tfrac{1}{3}}\epsilon, \quad (5.63a)$$

with

$$\epsilon = \frac{1}{2\sqrt{2}\mathcal{I}gF'(0)}. \quad (5.63b)$$

6. Further Developments

In this section, we discuss briefly some of more recent developments in the heavy-meson-soliton bound state approach to heavy baryons, in particular, the role of light vector mesons and finite mass corrections. Because of space limitation, we shall have to leave out various applications of this model, *e.g.*, the heavy-baryon Isgur–Wise function,⁷⁹ the $SU(3)$ extension of the background soliton,⁸⁰ exotic⁸¹ and excited⁷⁸ states of heavy baryons, etc.

6.1. Light Vector Mesons

So far we have considered interactions among the heavy mesons and the light pseudoscalar mesons to first order in derivative expansion involving the latter. It is known that in low-energy hadron physics, the vector mesons saturate the next-to-leading-order counter terms (dimension-four operators) in chiral Lagrangians and improve substantially the description of light meson and baryon dynamics. It is therefore natural to expect that introducing light vector mesons would have non-trivial effects on the interaction of heavy particles with light ones.^{82,83} For example, the semileptonic $D \rightarrow K^*$ transition appears to dominate over $D \rightarrow K\pi$.¹⁰ Indeed it has been recently shown that light vector mesons play an important role in heavy-meson-soliton bound states³³: the ρ -meson contribution to the binding energy is found to be 60% as large as that of the pion while the ω -meson contribution is 40% as large. The sign of the coupling constants involved is not yet determined and hence one cannot say whether their contributions are attractive or repulsive.^{80,82,83,84} In this section, we choose one possible case as an illustration. We consider the possibility that the ω contributes repulsively and ρ attractively³³ with the Lagrangian developed in Sec. 3. For the case that both ω and ρ mesons contribute attractively, see Ref. 80.

To proceed, we first modify Lagrangian (4.33) so as to incorporate the light vector mesons ρ and ω . One may do this [in the $SU(2)$ sector where there is no anomaly] either by the external-gauging of the flavor (the massive Yang-Mills method)⁸² or by the hidden local symmetry approach.⁸³ Here we follow Ref. 82 which uses the former approach. Let the vector and axial vector mesons be linear combinations of the fields A_μ^L and A_μ^R which transform under the chiral transformation Eqs. (2.14) and (2.17) as

$$A_\mu^L \rightarrow A'^L_\mu = L A_\mu^L L^\dagger, \quad A_\mu^R \rightarrow A'^R_\mu = R A_\mu^R R^\dagger. \quad (6.1)$$

We integrate out the axial vector mesons⁸⁵ by writing A_μ^L and A_μ^R in terms of the physical vector field $\rho_\mu [= \frac{1}{2}(\omega_\mu \mathbf{1} + \vec{\tau} \cdot \vec{\rho}_\mu)]$ ^{#22} as

$$\begin{aligned} A_\mu^L &= \xi \rho_\mu \xi^\dagger + \frac{i}{g_*} \xi \partial_\mu \xi^\dagger, \\ A_\mu^R &= \xi^\dagger \rho_\mu \xi + \frac{i}{g_*} \xi^\dagger \partial_\mu \xi, \end{aligned} \quad (6.2)$$

with the vector meson coupling constant g_* . One can see that ρ_μ transforms as

$$\rho_\mu \rightarrow \rho'_\mu = \vartheta \rho_\mu \vartheta^\dagger + \frac{i}{g_*} \vartheta \partial_\mu \vartheta^\dagger, \quad (6.3)$$

^{#22}Here, we include the isoscalar ω meson in the flavor symmetry group $U(2)$.

and its field strength tensor $F_{\mu\nu}(\rho) \equiv \partial_\mu \rho_\nu - \partial_\nu \rho_\mu - ig_*[\rho_\mu, \rho_\nu]$ as

$$F_{\mu\nu}(\rho) \rightarrow F'_{\mu\nu}(\rho) = \vartheta F_{\mu\nu}(\rho) \vartheta^\dagger. \quad (6.4)$$

Now, the “minimal” chiral Lagrangian of light pseudoscalars and vectors⁸⁵ corresponding to the normal sector of \mathcal{L}_M in the Lagrangian (4.33) is

$$\begin{aligned} \mathcal{L}'_M = & -\frac{1}{2} \text{Tr}[F_{\mu\nu}(\rho) F^{\mu\nu}(\rho)] + \frac{m_\rho^2}{4a} (1+a) \text{Tr}(A_\mu^L A^{\mu L} + A_\mu^R A^{\mu R}) \\ & - \frac{m_\rho^2}{2a} (1-a) \text{Tr}(A_\mu^L U A^{\mu R} U^\dagger), \end{aligned} \quad (6.5)$$

where m_ρ is the light vector meson mass and a is a dimensionless constant defined by $a = (m_\rho/f_\pi g_*)^2$. The “magic” value $a=2$ again gives the KSFR relation. Note that this Lagrangian contains the kinetic and interaction terms of the pseudoscalar mesons; *i.e.*, $\frac{1}{4} f_\pi^2 \text{Tr}(\partial_\mu U \partial^\mu U^\dagger)$, and that it is identical to the upper $SU(2)$ part of Eq. (3.13) in unitary gauge $\xi_L^\dagger = \xi_R = \xi$ in the hidden gauge symmetry approach; *viz.*,

$$\begin{aligned} & \frac{m_\rho^2}{4a} \text{Tr}(A_\mu^L A^{\mu L} + A_\mu^R A^{\mu R} \mp 2A_\mu^L U A^{\mu R} U^\dagger) \\ &= \frac{1}{4} f_\pi^2 g_*^2 \text{Tr}[\xi^\dagger A_\mu^L \xi \mp \xi A_\mu^R \xi^\dagger]^2 \\ &= \frac{1}{4} f_\pi^2 g_*^2 \text{Tr}[(\rho_\mu + \frac{i}{g_*} \partial_\mu \xi^\dagger \xi) \mp (\rho_\mu + \frac{i}{g_*} \partial_\mu \xi \xi^\dagger)]^2 \\ &= -\frac{1}{4} f_\pi^2 \text{Tr}[\mathcal{D}_\mu \xi^\dagger \xi \mp \mathcal{D}_\mu \xi \xi^\dagger]^2, \text{ with } \mathcal{D}_\mu \xi = (\partial_\mu - ig_* \rho_\mu) \xi. \end{aligned}$$

The anomalous-parity sector of \mathcal{L}_M for the light mesons is given by^{67,86}

$$\begin{aligned} \mathcal{L}'_{WZ} = & \frac{N_c g_*}{2} \omega_\mu \varepsilon^{\mu\nu\lambda\kappa} \frac{1}{24\pi^2} \text{Tr}(U^\dagger \partial_\nu U U^\dagger \partial_\lambda U U^\dagger \partial_\kappa U) \\ & + \frac{N_c g_*^2}{64\pi^2} \varepsilon^{\mu\nu\lambda\kappa} \omega_{\mu\nu} \text{Tr} \left\{ i\vec{\tau} \cdot \vec{\rho}_\lambda (U^\dagger \partial_\kappa U + \partial_\kappa U U^\dagger) + \frac{g_*}{2} \vec{\tau} \cdot \vec{\rho}_\lambda U^\dagger \vec{\tau} \cdot \vec{\rho}_\kappa U \right\}, \end{aligned} \quad (6.6)$$

with the number of colors $N_c(=3)$ and

$$\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu. \quad (6.6a)$$

The first term of \mathcal{L}'_{WZ} gives the ω -coupling to the topologically conserved baryon number current B^μ with the ωNN coupling⁸⁷ given by universality,

$$g_{\omega\pi\pi\pi} = g_{\omega NN} = g_\omega = \frac{1}{2} N_c g_* \cong 9 \quad (6.6b)$$

and the second term describes $\omega\rho\pi$ interactions.

The rest of the Lagrangian (4.33) can be generalized by replacing the covariant derivative D_μ appearing there by a suitably extended form. At this point, one has a choice. As can be seen from (2.16) and (6.3), both the “vector field” V_μ built of the

pseudoscalar fields, and the vector-meson field ρ_μ transform in the same way. Therefore, a generalized covariant derivative can be defined as

$$\begin{aligned} D'_\mu H &= H[\bar{\partial}_\mu + i\alpha g_* \rho_\mu^\dagger + (1 - \alpha)V_\mu^\dagger], \\ D'_\mu \bar{H} &= [\partial_\mu - i\alpha g_* \rho_\mu + (1 - \alpha)V_\mu]\bar{H}, \end{aligned} \quad (6.7)$$

with a dimensionless parameter α which reflects the extent to which the two pseudoscalars emitted in relative P -wave come from an intermediate vector state, with $\alpha = 1$ representing “vector-meson dominance.” Chiral symmetry alone cannot fix the value of α and one has to fix it from experiments.

Another chiral-invariant interaction that may be important is

$$\mathcal{L}_{PP^*\rho} = i\beta \text{Tr}[H\gamma^\mu\gamma^\nu F_{\mu\nu}(\rho)\bar{H}], \quad (6.8)$$

where β is an unknown coupling constant. This Lagrangian can be gotten from the heavy-mass limit of the ordinary Lagrangian given in terms of P and P_μ^* :

$$\mathcal{L}_{PP^*\rho} = -2\beta_1 P_\mu^* F^{\mu\nu}(\rho) P_\nu^{*\dagger} - \beta_2 \varepsilon^{\mu\nu\lambda\kappa} (P F_{\mu\nu} P_{\lambda\kappa}^{*\dagger} + P_{\lambda\kappa}^* F_{\mu\nu} P^\dagger).$$

The heavy quark symmetry implies $\beta_1/M_{P^*} = \beta_2 = 2\beta$. This higher-dimension term is needed to provide the trilinear coupling among D , D_s^* and K^* (for three light flavors),^{82,83} which gives the D_s^* pole contribution to $D \rightarrow K^*$ decay. In Ref. 83, the trilinear coupling constant β is determined by comparing the calculated $D \rightarrow K^*$ semileptonic weak decay with the data from the E653 Collaboration⁸⁸

$$\beta/g_* = -(0.63 \pm 0.17)\text{GeV}^{-1}. \quad (6.9)$$

One may add a term similar to (6.8) in which $F_{\mu\nu}(\rho)$ is replaced by $F_{\mu\nu}(V) = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]$:

$$\mathcal{L}'_{PP^*\rho} = i\beta' \text{Tr}[H\gamma^\mu\gamma^\nu F_{\mu\nu}(V)\bar{H}].$$

In the spirit of (light) vector-meson dominance, we would expect this term to be less important.

To sum up, the leading terms of the chiral invariant heavy-meson Lagrangian written in terms of the doublet H field are

$$\mathcal{L} = \mathcal{L}'_M + \mathcal{L}'_{WZ} - iv^\mu \text{Tr}(D'_\mu H \bar{H}) - g \text{Tr}(H \gamma_5 A_\mu \gamma^\mu \bar{H}) + i\beta \text{Tr}(H \gamma^\mu \gamma^\nu F_{\mu\nu}(\rho) \bar{H}). \quad (6.10)$$

This Lagrangian contains two parameters, α and β , in addition to the coupling constant g . For simplicity, we have not incorporated the chiral symmetry breaking pion mass term.

The Lagrangian $\mathcal{L}'_M + \mathcal{L}'_{WZ}$ for the light mesons supports a classical soliton solution with the hedgehog ansatz

$$\begin{aligned} U_c(\vec{r}) &= \exp[i\vec{\tau} \cdot \hat{r} F(r)], \\ \omega_c^{\mu=0}(\vec{r}) &= \omega(r), \quad \omega_c^{\mu=i}(\vec{r}) = 0, \\ \rho_c^{\mu=i,a}(\vec{r}) &= \frac{1}{g_* r} \varepsilon^{ika} \hat{r}^k G(r), \quad \rho_c^{\mu=0,a}(\vec{r}) = 0, \end{aligned} \quad (6.11)$$

with three radial functions: $F(r)$, $G(r)$ and $\omega(r)$. Here, $i=1,2,3$ denote the three space components of the vector fields and $a=1,2,3$ are the isospin indices. Substituting the hedgehog ansatz (6.11) into the Lagrangian $\mathcal{L}'_M + \mathcal{L}'_{WZ}$, one gets the static energy functional for the soliton mass

$$E[F, G, \omega] = 4\pi \int_0^\infty r^2 dr \left\{ \frac{f_\pi^2}{2} \left[F'^2 + \frac{2 \sin^2 F}{r^2} \right] + \frac{4f_\pi^2}{r^2} (G + 1 - \cos F)^2 \right. \\ \left. - (f_\pi^2 g_*^2 \omega^2 + \frac{1}{2} \omega'^2) + \frac{G'^2}{g_*^2 r^2} + \frac{G^2 (G + 2)^2}{2g_*^2 r^4} \right. \\ \left. + \frac{3g_*}{4\pi^2} \omega F' \frac{\sin^2 F}{r^2} + \frac{3g}{16\pi^2} \omega' \frac{G(G + 2)}{r^2} \sin 2F \right\}. \quad (6.12)$$

Coupled equations of motion for $F(r)$, $G(r)$ and $\omega(r)$ can be derived by functionally minimizing the static energy:

$$F'' = -\frac{2}{r} F' + \frac{1}{r^2} [4(G + 1) \sin F - \sin 2F] \\ + \frac{3g_*}{8\pi^2 f_\pi^2} \frac{\omega'}{r^2} [-2 \sin^2 F + G(G + 2) \cos 2F], \\ G'' = 2g_*^2 f_\pi^2 (G + 1 - \cos F) + \frac{1}{r^2} G(G + 1)(G + 2) \\ + \frac{3g_*^3}{16\pi^2} \omega' (G + 1) \sin 2F, \\ \omega'' = -\frac{2}{r} \omega' + 2f_\pi^2 g_*^2 \omega - \frac{3g_*}{4\pi^2 r^2} F' \sin^2 F \\ + \frac{3g_*}{8\pi^2 r^2} [G(G + 2) F' \cos 2F + G'(G + 1) \sin 2F]. \quad (6.13)$$

To ensure a singularity-free baryon number $B = n$ solution and finiteness of the energy, we impose the boundary conditions on $F(r)$, $G(r)$ and $\omega(r)$:

$$F(0) = n\pi, \quad G(0) = -[1 - (-1)^n], \quad \omega'(0) = 0, \\ F(\infty) = 0, \quad G(\infty) = 0, \quad \omega(\infty) = 0. \quad (6.14)$$

The stability of the soliton solution is assured by the repulsion generated by the vector mesons at short distance (without any additional term like the Skyrme term).

Classical eigenmodes of the heavy mesons moving under the static potentials provided by the classical soliton configuration sitting at the origin are given by the equation of motion

$$i\partial_0 h(\vec{r}, t) = h(\vec{r}, t) \left\{ g\vec{\sigma} \cdot \vec{A} + \frac{1}{2} \alpha g_* \omega(r) + i\beta \gamma^i \gamma^j F^{ij}(\rho) \right\}, \quad (6.15)$$

where

$$\vec{A} = \frac{1}{2} \left[\frac{\sin F}{r} \vec{\tau} + (F' - \frac{\sin F}{r}) \hat{r} \vec{\tau} \cdot \hat{r} \right], \quad (6.15a)$$

$$\gamma^i \gamma^j F^{ij}(\rho) = \frac{i}{g_* r} \left[-G' \vec{\sigma} \cdot \vec{\tau} + \frac{1}{r} (rG' - G(G+2)) \vec{\sigma} \cdot \hat{r} \vec{\tau} \cdot \hat{r} \right]. \quad (6.15b)$$

Near the origin where the heavy mesons are expected to be strongly peaked, the profile functions have the asymptotic structures

$$\begin{aligned} F(r) &= \pi + rF'(0) + \frac{1}{6}r^3 F'''(0) + \dots, \\ \omega(r) &= \omega(0) + \frac{1}{2}r^2 \omega''(0) + \dots, \\ G(r) &= -2 + \frac{1}{2}r^2 G''(0) + \frac{1}{24}r^4 G''''(0) + \dots, \end{aligned} \quad (6.16)$$

so that the potentials can be expanded as

$$\begin{aligned} \omega &= \omega(0) + O(r^2), \\ \vec{\sigma} \cdot \vec{A} &= \frac{1}{2}F'(0)(2\vec{\sigma} \cdot \hat{r} \vec{\tau} \cdot \hat{r} - \vec{\sigma} \cdot \vec{\tau}) + O(r^2), \\ \gamma^i \gamma^j F^{ij}(\rho) &= \frac{i}{g_*} G''(0)(2\vec{\sigma} \cdot \hat{r} \vec{\tau} \cdot \hat{r} - \vec{\sigma} \cdot \vec{\tau}) + O(r^2). \end{aligned} \quad (6.17)$$

Note that $\vec{\sigma} \cdot \vec{A}$ and $\gamma^i \gamma^j F^{ij}(\rho)$ have the same structure at the origin. The problem becomes now exactly the same as that in Sec. 5.1 except that we have an overall energy shift by an amount of $\frac{1}{2}\alpha\omega(0)$ and that $\frac{1}{2}gF'(0)$ appearing there is now replaced by $\frac{1}{2}gF' - \beta G''(0)/g_*$: the eigenenergy of the bound state is

$$\varepsilon_{bs} = -\frac{3}{2}gF'(0) + 3\frac{\beta}{g_*}G''(0) - \frac{1}{2}\alpha g_* \omega(0), \quad (6.18)$$

while the corresponding eigenfunction remains the same as given in Table 5.2. As for the other two “unbound” states, the g and β terms are both multiplied by $-\frac{1}{3}$ while the α term remains unchanged.

Now, our problem is to calculate the derivative of the radial functions $F(r)$, $G(r)$ and $\omega(r)$ that figure in the bound state energy (6.18). These are found by solving the coupled differential equations (6.13) subject to the boundary conditions (6.14). The solution depends on the structure and parameters of the light meson Lagrangian.

Instead of going into details, we shall be content with a rough estimate of the contribution of the vector mesons to the heavy meson binding energy to see their nontrivial roles. In the limit $m_\rho \rightarrow \infty$, ρ_μ is constrained to

$$\rho_\mu = \frac{i}{g_*} V_\mu, \quad (6.19)$$

which gives

$$G(r) = -1 + \cos F(r), \quad (6.19a)$$

and consequently

$$G''(0) = [F'(0)]^2. \quad (6.19b)$$

As for the ω -meson radial function $\omega(r)$, it can be integrated to give

$$\omega(r) = \int_0^\infty r'^2 dr' G(r, r') \left[\frac{3g_*}{4\pi^2} \frac{\sin^2 F(r')}{r'^2} F'(r') \right], \quad (6.20)$$

with the help of the radial Green's function

$$G(r, r') = \frac{e^{-m_\omega|r-r'|} - e^{-m_\omega(r+r')}}{2m_\omega r r'}. \quad (6.20a)$$

Here, we have neglected the ω - ρ coupling term. To leading order in m_ω , Eq. (6.20) becomes

$$\begin{aligned} \omega(0) &= \int_0^\infty dr' \left[\frac{e^{-m_\omega r'}}{r'} \right] \left[\frac{3g_*}{4\pi^2} \frac{\sin^2 F(r')}{r'^2} F'(r') \right] \\ &\sim \frac{3g_*}{m_\omega^2 4\pi^2} [F'(0)]^3. \end{aligned} \quad (6.20b)$$

Substituting Eqs. (6.19b) and (6.20b) into Eq. (6.18) with $F'(0) \sim -0.7$ GeV yields

$$\varepsilon_{bs} = 1.05g + 1.47 \frac{\beta}{g_*} + 0.75\alpha. \quad (6.21)$$

Examining the individual terms with $g = -\frac{3}{4}$, $\beta/g_* = -0.67$ and $\alpha = 1$ (vector meson dominance), we see that the “attractive”^{#23} ρ -meson and “repulsive”^{#23} ω -meson contributions are comparable to that of the pion. Of course this is just a rough estimate but seems to represent a typical situation. It is clear that the vector mesons and the higher-order derivative terms are expected to play a nontrivial role.

6.2. Finite Mass Corrections

So far, we have limited ourselves to the heavy-quark limit. Thus heavy-baryon properties have been computed to leading order in $1/m_Q$, that is to $O(1)$. In the heavy-quark limit, Σ_Q and Σ_Q^* are in a degenerate multiplet with isospin one and spin 1/2 and 3/2, respectively. The (hyperfine) mass splitting between these states is an $1/m_Q$ effect.

The $\Sigma_Q^* - \Sigma_Q$ mass difference due to the leading heavy-quark symmetry breaking was first computed in Ref. 31. The leading order Lagrangian in derivative expansion that breaks the heavy quark symmetry is³⁶

$$\mathcal{L}_1 = \frac{\lambda_2}{m_Q} \text{Tr} \sigma^{\mu\nu} H \sigma_{\mu\nu} \bar{H}. \quad (6.22)$$

^{#23}One should perhaps not conclude that the effect of the ρ -meson is truly repulsive and that of ω -meson truly attractive. First of all, their effects are strongly dependent on the coupling constant β/g_* and the parameter α . In Ref. 84, β/g_* is found to have a positive sign and in Ref. 33 $\alpha = -2$ is taken for a better fit. Furthermore, the vector mesons contribute to the energy of the heavy meson bound state indirectly by changing the pion profile $F(r)$. Note also that unless stabilized by vector mesons the soliton shrinks to zero size, $F'(0) \rightarrow -\infty$.

This gives a $P_Q^*-P_Q$ mass difference of $-8\lambda_2/m_Q$. One can easily check that it is the next-to-leading order term ignored in obtaining Eq. (4.16) by substituting Eq. (4.15) into Lagrangian (2.19): *viz.*,

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{2\bar{m}_P} \left\{ (\bar{m}_P^2 - m_P^2) P_v P_v^\dagger - (\bar{m}_P^2 - m_{P^*}^2) P_v^{*\mu} P_{v\mu}^{*\dagger} \right\} \\ &= \frac{1}{4}(m_{P^*} - m_P) \left\{ 3P_v P_v^\dagger + P_v^{*\mu} P_{v\mu}^{*\dagger} \right\}.\end{aligned}$$

Here we shall follow Ref. 31 and use the alternative bound-state approach described in Sec. 5.3 to calculate the $\Sigma_Q^*-\Sigma_Q$. The soliton-meson states $|1, 1; N\rangle$ and $|1, 1; \Delta\rangle$ appearing in Eq. (5.63) can be written explicitly in terms of the soliton states and the spin of the light degrees of freedom of H ,

$$\begin{aligned}|1, 1, 1; N\rangle &= |\tfrac{1}{2}\rangle_N |\uparrow\rangle_\ell, \\ |1, 1, 0; N\rangle &= \sqrt{\tfrac{1}{2}}|\tfrac{1}{2}\rangle_N |\downarrow\rangle_\ell + \sqrt{\tfrac{1}{2}}|-\tfrac{1}{2}\rangle_N |\uparrow\rangle_\ell, \\ |0, 0, 0; N\rangle &= \sqrt{\tfrac{1}{2}}|\tfrac{1}{2}\rangle_N |\downarrow\rangle_\ell - \sqrt{\tfrac{1}{2}}|-\tfrac{1}{2}\rangle_N |\uparrow\rangle_\ell, \\ |1, 1, 1; \Delta\rangle &= \sqrt{\tfrac{3}{4}}|\tfrac{3}{2}\rangle_\Delta |\downarrow\rangle_\ell - \sqrt{\tfrac{1}{4}}|\tfrac{1}{2}\rangle_\Delta |\uparrow\rangle_\ell, \\ |1, 1, 0; \Delta\rangle &= \sqrt{\tfrac{1}{2}}|\tfrac{1}{2}\rangle_\Delta |\downarrow\rangle_\ell - \sqrt{\tfrac{1}{4}}|-\tfrac{1}{2}\rangle_\Delta |\uparrow\rangle_\ell,\end{aligned}\tag{6.23}$$

where $|\rangle_\ell$ is the spin state of the light degrees of freedom in the heavy meson, $|m\rangle_{N,\Delta}$ is the soliton state in the N or Δ sector with $s_{\ell 3} = m$, and $|i, j_\ell, m; N, \Delta\rangle$ is the bound state $|i, j_\ell; N, \Delta\rangle$ with $j_{\ell 3} = m$. Furthermore, the tensor product of the light degrees of freedom and heavy quark in H can be re-expressed in terms of P and P^* mesons,

$$\begin{aligned}|\uparrow\rangle_\ell |\uparrow\rangle_Q &= |P_Q^*, 1\rangle, \\ |\downarrow\rangle_\ell |\downarrow\rangle_Q &= |P_Q^*, -1\rangle, \\ |\uparrow\rangle_\ell |\downarrow\rangle_Q &= \sqrt{\tfrac{1}{2}}|P_Q\rangle + \sqrt{\tfrac{1}{2}}|P_Q^*, 0\rangle, \\ |\downarrow\rangle_\ell |\uparrow\rangle_Q &= \sqrt{\tfrac{1}{2}}|P_Q\rangle - \sqrt{\tfrac{1}{2}}|P_Q^*, 0\rangle,\end{aligned}\tag{6.24}$$

where $|P_Q^*, m\rangle$ is the P_Q^* meson with $s_3 = m$. Finally, the Σ_Q^* , Σ_Q and Λ_Q states are written explicitly as

$$\begin{aligned}|\Sigma_Q^*, \tfrac{3}{2}\rangle &= -\sqrt{\tfrac{3}{8}}b|\tfrac{3}{2}\rangle_\Delta |P_Q\rangle - \tfrac{1}{2}b|\tfrac{1}{2}\rangle_\Delta |P_Q^*, 1\rangle + a|\tfrac{1}{2}\rangle_N |P_Q^*, 1\rangle + \sqrt{\tfrac{3}{8}}b|\tfrac{3}{2}\rangle_\Delta |P_Q^*, 0\rangle, \\ |\Sigma_Q, \tfrac{1}{2}\rangle &= \sqrt{\tfrac{3}{4}}a|\tfrac{1}{2}\rangle_N |P_Q\rangle + \sqrt{\tfrac{1}{2}}b|\tfrac{3}{2}\rangle_\Delta |P_Q^*, -1\rangle + \sqrt{\tfrac{1}{6}}b|-\tfrac{1}{2}\rangle_\Delta |P_Q^*, 1\rangle \\ &\quad - \sqrt{\tfrac{1}{6}}a|-\tfrac{1}{2}\rangle_N |P_Q^*, 1\rangle - \sqrt{\tfrac{1}{3}}b|\tfrac{1}{2}\rangle_\Delta |P_Q^*, 0\rangle + \sqrt{\tfrac{1}{12}}a|\tfrac{1}{2}\rangle_N |P_Q^*, 0\rangle, \\ |\Lambda_Q, \tfrac{1}{2}\rangle &= \tfrac{1}{2}|\tfrac{1}{2}\rangle_N |P_Q^*, 0\rangle - \tfrac{1}{2}|\tfrac{1}{2}\rangle_N |P_Q\rangle - \sqrt{\tfrac{1}{2}}|-\tfrac{1}{2}\rangle_N |P_Q^*, 1\rangle.\end{aligned}\tag{6.25}$$

Now, the Σ_Q^* , Σ_Q and Λ_Q masses including the next-to-leading order Lagrangian can be read off directly from the expressions for the states (6.25) :

$$\begin{aligned} m_{\Sigma_Q^*} &= -\frac{3}{2}gF'(0) + a^2m_N + b^2m_\Delta + \frac{3}{8}b^2m_P + (a^2 + \frac{5}{8}b^2)m_{P^*}, \\ m_{\Sigma_Q} &= -\frac{3}{2}gF'(0) + a^2m_N + b^2m_\Delta + \frac{3}{4}a^2m_P + (\frac{1}{4}a^2 + b^2)m_{P^*}, \\ m_{\Lambda_Q} &= -\frac{3}{2}gF'(0) + m_N + \frac{1}{4}m_P + \frac{3}{4}m_{P^*}, \end{aligned} \quad (6.26)$$

where the coefficient a^2 is the probability that Σ_Q^* contains a nucleon, $3b^2/8$ is the probability that Σ_Q^* contains a P_Q , etc. The Σ_Q^* - Σ_Q mass difference is obtained as

$$m_{\Sigma_Q^*} - m_{\Sigma_Q} = \frac{3}{8}(2a^2 - b^2)(m_{P^*} - m_P) = \frac{(m_\Delta - m_N)(m_{P^*} - m_P)}{4gF'(0)}. \quad (6.27)$$

Note that the mass splittings have the dependence on m_Q and N_c that agrees with the constituent quark model. The P^* - P mass difference is of order $1/m_Q$ and the Δ - N mass difference is of order $1/N_c$. This implies that the Σ_Q^* - Σ mass difference is of order $1/(m_Q N_c)$. Substituting $gF'(0) = 419$ MeV, we obtain

$$m_{\Sigma_c^*} - m_{\Sigma_c} = 25 \text{ MeV} \quad \text{and} \quad m_{\Sigma_b^*} - m_{\Sigma_b} = 8 \text{ MeV}. \quad (6.28)$$

The experimentally measured Σ_c^* - Σ_c mass difference ~ 77 MeV is three times as big as this Skyrme model prediction and, compared to the quark model prediction, they are reduced by a factor two. Note however, they are not the *whole story* of the large N_c prediction. For example, the Lagrangian

$$\mathcal{L}_2 = \frac{\lambda_\pi}{m_Q} \text{Tr}(H A_\mu \bar{H} \gamma^\mu \gamma_5), \quad (6.29)$$

also breaks the heavy quark symmetry, and upsets the relation between the $P^*P^*\pi$ and $P^*P\pi$ coupling constants.

For an illustration of the equivalence of the two approaches discussed in this paper, we now do the same calculation in the CK bound-state approach described in Sec. 5.1 and Sec. 5.2. With the heavy quark symmetry breaking Lagrangian \mathcal{L}_1 , the equation of motion for the heavy meson gets an additional term

$$i\partial_0 h(\vec{r}, t) = \varepsilon h(\vec{r}, t) = h(\vec{r}, t)[g\vec{A} \cdot \vec{\sigma}] + \frac{2\lambda_2}{m_Q} \vec{\sigma} \cdot h(\vec{r}, t) \vec{\sigma}. \quad (6.30)$$

We shall now take the last term as a perturbation and compute its effect on the $k = 1/2$ bound state obtained in Sec. 5.1. The last term breaks only the heavy quark spin symmetry. The grand spin is still a good symmetry of the equation of motion so that the eigenstates can be classified by the corresponding quantum number. Assuming the same radial functions peaked strongly at the origin as in Sec. 5.1, the problem reduces to finding the eigenfunction of the equation

$$\varepsilon \mathcal{K}_{\frac{1}{2}k_3} = \frac{1}{2}gF'(0)\mathcal{K}_{\frac{1}{2}k_3}(\vec{\tau} \cdot \hat{r})[\vec{\sigma} \cdot \vec{\tau}](\vec{\tau} \cdot \hat{r}) + \frac{2\lambda_2}{m_Q} \vec{\sigma} \cdot \mathcal{K}_{\frac{1}{2}k_3} \vec{\sigma}. \quad (6.31)$$

The eigenstates $\mathcal{K}_{\frac{1}{2}k_3}$ can be expanded in terms of the three possible basis states $\mathcal{K}_{\frac{1}{2}k_3}^{(i)}$ as in Eq. (5.20) with the expansion coefficients given by the solution of the secular equation

$$\sum_{j=1}^3 \left(\mathcal{M}_{ij}^0 + \mathcal{M}_{ij}^1 \right) c_j = -\varepsilon c_i, \quad (6.32)$$

with the matrix elements M_{ij}^0 and M_{ij}^1 defined by

$$M_{ij}^0 = \int d\Omega \text{Tr} \{ \mathcal{K}_{\frac{1}{2}k_3}^{(i)} (\vec{\tau} \cdot \hat{r}) [\frac{1}{2} g F'(0) (\vec{\sigma} \cdot \vec{\tau})] (\vec{\tau} \cdot \hat{r}) \bar{\mathcal{K}}_{\frac{1}{2}k_3}^{(j)} \}, \quad (6.32a)$$

$$M_{ij}^1 = \frac{2\lambda_2}{m_Q} \int d\Omega \text{Tr} \{ \vec{\sigma} \cdot \mathcal{K}_{\frac{1}{2}k_3}^{(i)} \vec{\sigma} \bar{\mathcal{K}}_{\frac{1}{2}k_3}^{(j)} \}. \quad (6.32b)$$

The matrix elements M_{ij}^0 were evaluated in Sec. 5.1. For M_{ij}^1 , we exploit the fact that $\frac{1}{2}\vec{\sigma}$ acting on the right hand side of \mathcal{K} is the heavy quark spin operator while $-\frac{1}{2}\vec{\sigma}$ acting on its left hand side is the light quark spin operator of the heavy mesons. Actually, $\mathcal{K}_{\frac{1}{2}k_3}^{(i)}$ ($i = 1, 2, 3$) are the eigenstates of the operator:

$$\vec{\sigma} \cdot \mathcal{K}_{\frac{1}{2}k_3}^{(i)} \vec{\sigma} = \{-4\vec{S}_Q \cdot \vec{S}_\ell\} \mathcal{K}_{\frac{1}{2}k_3}^{(i)} = -2(s(s+1) - 3/2) \mathcal{K}_{\frac{1}{2}k_3}^{(i)} = \begin{cases} 3\mathcal{K}_{\frac{1}{2}k_3}^{(i)} & i = 1 \\ -\mathcal{K}_{\frac{1}{2}k_3}^{(i)} & i = 2, 3 \end{cases}. \quad (6.33)$$

The result is

$$M^1 = -\frac{2\lambda_2}{m_Q} \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (6.34)$$

Up to first order in perturbation, the bound state energy remains unchanged: $\varepsilon = -\frac{3}{2}gF'(0)$. On the other hand, the corresponding eigenstate $\mathcal{K}_{\frac{1}{2}k_3}^{bs}$ is perturbed to

$$\mathcal{K}_{\frac{1}{2}k_3}^{bs} = \frac{1}{2}(1 + 3\epsilon)\mathcal{K}_{\frac{1}{2}k_3}^{(1)} - \frac{\sqrt{3}}{2}(1 - \epsilon)\mathcal{K}_{\frac{1}{2}k_3}^{(2)}, \quad (6.35)$$

with

$$\epsilon = -\frac{\lambda_2}{m_Q} \frac{1}{gF'(0)}. \quad (6.36)$$

With $\epsilon = 0$, the eigenstate returns to that in the heavy quark limit.

The heavy baryons can be obtained by quantizing the soliton-heavy meson bound state in the same way as explained in Sec. 5.2. It leads to the heavy baryon states of Eq. (5.37) with $|m\rangle_{bs}$ replaced by the perturbed state of Eq. (6.35). Due to the perturbation, the expectation value of $\vec{\Phi}(\infty)$ defined by Eq. (5.29b) with respect to the bound states *does not vanish*. According to Wigner-Eckart theorem, the expectation value can be expressed as

$${}_{bs}\langle m' | \vec{\Phi}(\infty) | m \rangle_{bs} = -c \langle \frac{1}{2}m' | \vec{K} | \frac{1}{2}m \rangle, \quad (6.37)$$

where the “hyperfine constant” is given by

$$c = 2\epsilon = -\frac{2\lambda_2}{m_Q} \frac{1}{gF'(0)}. \quad (6.38)$$

With the help of Eq. (6.37), we can compute the expectation value of the collective Hamiltonian (5.29)

$$E_{i,j} = M_{sol} + \varepsilon_{bs} + \frac{1}{2\mathcal{I}} \{(1-c)i(i+1) + cj(j+1) - ck(k+1) + \frac{3}{4}\}. \quad (6.39)$$

Explicitly, we have the heavy baryon masses

$$\begin{aligned} m_{\Lambda_Q} &= M_{sol} + \overline{m}_P - \frac{3}{2}gF'(0) + 3/8\mathcal{I}, \\ m_{\Sigma_Q} &= M_{sol} + \overline{m}_P - \frac{3}{2}gF'(0) + (11-8c)/8\mathcal{I}, \\ m_{\Sigma_Q^*} &= M_{sol} + \overline{m}_P - \frac{3}{2}gF'(0) + (11+4c)/8\mathcal{I}. \end{aligned} \quad (6.40)$$

We see that these are identical to Eq. (6.26).

One more important aspect of the bound state approach with finite mass heavy mesons has to do with the turning-on of the kinetic motions. In the heavy quark limit, the heavy mesons are approximated to sit at the origin of the soliton and the kinetic motion of the heavy mesons is neglected. As the heavy meson masses become finite, the effects of the kinetic motion increase. For example, the binding energy of the heavy mesons would be reduced compared with that in the infinite mass limit. Such $1/m_Q$ corrections have been studied in Ref. 37.

Let us now return to the Lagrangian (2.19) written in terms of the component heavy-meson fields P and P_μ^* . The equations of motion are

$$\begin{aligned} (D_\mu D^\mu + m_P^2)P &= f_Q P_\mu^* A^\mu, \\ D_\mu P^{*\mu\nu} + m_{P^*}^2 P^{*\nu} &= -f_Q P A^\nu + g_Q \varepsilon^{\mu\nu\lambda\rho} P_{\mu\rho}^* A_\lambda. \end{aligned} \quad (6.41)$$

The momenta conjugate to the meson fields P and P_μ^* , respectively, are

$$\begin{aligned} \Pi &= \frac{\partial \mathcal{L}}{\partial \dot{P}} = D_0 P^\dagger, \\ \Pi^{*i} &= \frac{\partial \mathcal{L}}{\partial \dot{P}_i^*} = (P^{*i0})^\dagger - g_Q \epsilon^{ijk} A_j P_k^{*\dagger}, \end{aligned} \quad (6.42)$$

and similarly for Π^\dagger and $\Pi^{*i\dagger}$. Since Π_0^* vanishes identically, P_0^* is not an independent dynamical variable; it can be eliminated by using Eq. (6.41)

$$P^{*0} = -\frac{1}{m_{P^*}^2} (D_i \Pi^{*i\dagger} + \frac{1}{2} g_Q \epsilon^{ijk} P_{ij}^* A_k), \quad (6.43)$$

which results in a set of coupled equations

$$\begin{aligned}\dot{\vec{P}}^* &= -\vec{\Pi}^{*\dagger} + g_Q \vec{P}^* \times \vec{A} + \frac{1}{m_{P^*}^2} \vec{D} (\vec{D} \cdot \vec{\Pi}^{*\dagger}) + \frac{g_Q}{m_{P^*}^2} \vec{D} ((\vec{D} \times \vec{P}^*) \cdot \vec{A}), \\ \dot{\vec{\Pi}}^{*\dagger} &= \vec{D} \times (\vec{D} \times \vec{P}^*) + m_{P^*}^2 \vec{P}^* + f_Q P \vec{A} + g_Q \vec{\Pi}^{*\dagger} \times \vec{A} - g_Q^2 (\vec{P}^* \times \vec{A}) \times \vec{A} \\ &\quad - \frac{2g_Q}{m_{P^*}^2} \vec{D} \{ \vec{D} \cdot \vec{\Pi}^{*\dagger} + g_Q (\vec{D} \times \vec{P}^*) \cdot \vec{A} \} \times \vec{A}.\end{aligned}\quad (6.44)$$

where $\vec{D}P \equiv \vec{\nabla}P - P\vec{V}^\dagger$.

In order to express the equations of motion only in terms of P and \vec{P}^* , we use the fact that P_0^* field is at most of order $1/m_Q$; *viz.*,

$$P^{*0} \sim \frac{1}{m_{P^*}^2} D_i \dot{P}^{*i} = O(1/m_{P^*}). \quad (6.45)$$

Keeping this leading order term, we can express the equations of motion as

$$\ddot{\vec{P}}^* = +2g_Q \dot{\vec{P}}^* \times \vec{A} - \vec{D} \times (\vec{D} \times \vec{P}^*) - M_{P^*}^2 \vec{P}^* - f_Q P \vec{A} + \vec{D} (\vec{D} \cdot \vec{P}^*). \quad (6.46)$$

The eigenstates are classified by the grand spin quantum numbers, k , k_3 and the parity π . The wavefunctions of the $k^\pi = \frac{1}{2}^+$ state of our interest are expanded as

$$\begin{aligned}P_{\frac{1}{2}^+ k_3}(\vec{r}, t) &= e^{-i\omega t} \varphi(r) \mathcal{Y}_{\frac{1}{2}^+ k_3}(\hat{r}), \\ \vec{P}_{\frac{1}{2}^+ k_3}^*(\vec{r}, t) &= e^{-i\omega t} \sum_{\kappa} \varphi_{\kappa}^*(r) \vec{\mathcal{Y}}_{\frac{1}{2}^+ k_3}^{(\kappa)}(\hat{r}),\end{aligned}\quad (6.47)$$

where $\mathcal{Y}_{k^\pi k_3}$ and $\vec{\mathcal{Y}}_{k^\pi k_3}$ are the generalized spherical spinors and vector harmonics, respectively, and κ is an index to label the possible vector spherical harmonics with the same k , k_3 and parity π . These spherical harmonics correspond to $\mathcal{K}_{kk_3}^{(i)}$ in Sec. 5 with ω equal to $\overline{m}_P + \varepsilon$.

Since the spin of the heavy mesons is represented in a different way from the 4×4 matrix representation in the heavy quark limit, the explicit forms of \mathcal{Y}_{kk_3} and $\vec{\mathcal{Y}}_{kk_3}$ are different from those of $\mathcal{K}_{kk_3}^{(i)}$. As for the pseudoscalar meson, we have only one spherical spinor harmonics with $k^\pi = \frac{1}{2}^+$:

$$\mathcal{Y}_{\frac{1}{2}^+ \pm 1/2}(\hat{r}) = \frac{1}{\sqrt{4\pi}} \tilde{\phi}_{\pm} \vec{\tau} \cdot \hat{r}. \quad (6.48)$$

Here, $\tilde{\phi}_{\pm}$ is the same isospin basis for the heavy meson anti-doublets that we have used in Sec. 5. For vector mesons with spin 1, we can construct two different $k^P = \frac{1}{2}^+$ vector

spherical harmonics ^{#24}: viz.,

$$\begin{aligned}\vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(1)}(\hat{r}) &= \frac{1}{\sqrt{4\pi}} \tilde{\phi}_{\pm} \hat{r}, \\ \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(2)}(\hat{r}) &= i \frac{1}{\sqrt{8\pi}} \tilde{\phi}_{\pm} (\vec{\tau} \times \hat{r}).\end{aligned}\tag{6.49}$$

Putting

$$\begin{aligned}P(\vec{r}, t) &= e^{-i\omega t} \varphi(r) \mathcal{Y}_{\frac{1}{2}^+ \pm \frac{1}{2}}(\hat{r}), \\ \vec{\Phi}^*(\vec{r}, t) &= e^{-i\omega t} \left\{ \varphi_1^*(r) \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(1)}(\hat{r}) + \varphi_2^*(r) \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(2)}(\hat{r}) \right\},\end{aligned}\tag{6.50}$$

into the equations of motion (6.41) and (6.46), we obtain three coupled differential equations for the radial functions:

$$\begin{aligned}\varphi'' + \frac{2}{r} \varphi' + (\omega^2 - m_P^2 - \frac{2}{r^2}) \varphi &= 2v(v - \frac{2}{r}) \varphi + \frac{f_Q}{2} (a_1 + a_2) \varphi_1^* - \frac{1}{\sqrt{2}} f_Q a_1 \varphi_2^*, \\ \varphi_1^{*''} + \frac{2}{r} \varphi_1^{*'} + (\omega^2 - m_{P^*}^2 - \frac{2}{r^2}) \varphi_1^* &= \frac{f_Q}{2} (a_1 + a_2) \varphi + 2v^2 \varphi_1^* \\ &\quad + \sqrt{2} (g_Q a_1 \omega - \frac{1}{r} v + v') \varphi_2^*, \\ \varphi_2^{*''} + \frac{2}{r} \varphi_2^{*'} + (\omega^2 - m_{P^*}^2 - \frac{2}{r^2}) \varphi_2^* &= -\frac{f_Q}{\sqrt{2}} a_1 \varphi + \sqrt{2} (\omega g_Q a_1 - \frac{1}{r} v + v') \varphi_1^* \\ &\quad + (-\omega g_Q (a_1 + a_2) - \frac{4}{r} v + 4v^2) \varphi_2^*.\end{aligned}\tag{6.51}$$

The wavefunctions are normalized such that each mode carries one corresponding heavy flavor number:

$$1 = \int_0^\infty r^2 dr \left\{ 2\omega \left[|\varphi|^2 + |\varphi_1^*|^2 + |\varphi_2^*|^2 \right] + g_Q \left[(a_1 + a_2) |\varphi_2^*|^2 - \sqrt{2} a_1 (\varphi_1^{*\dagger} \varphi_2^* + \varphi_2^{*\dagger} \varphi_1^*) \right] \right\},\tag{6.52}$$

where we have kept terms up to next-to-leading order in $1/m_Q$.

^{#24}Here, we combine first the spin and orbital angular momentum bases to the total spin ($\vec{J} = \vec{S} + \vec{L}$) basis and then combine the spin and isospin. Then, $\vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(1)}(\hat{r})$ and $\vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(2)}(\hat{r})$ correspond to $J = 0$ and $J = 1$ states, respectively. This procedure enables us to proceed with the simple combinations such as $\vec{\tau} \cdot \hat{r}$ and $\vec{\tau} \times \hat{r}$ in the forthcoming calculations. One may obtain the vector spherical harmonics in different ways; for example, by combining first the isospin and orbital angular momentum to the $\vec{\Lambda} (= \vec{I} + \vec{L})$ basis and then to the spin basis, leading to

$$\begin{aligned}\vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(1)}(\hat{r}) &= \frac{1}{\sqrt{12\pi}} \vec{\tau} \cdot \hat{r} \vec{\tau} \chi_{\pm} = \sqrt{\frac{1}{3}} \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(1)} + \sqrt{\frac{2}{3}} \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(2)}, \\ \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(2)}(\hat{r}) &= \frac{1}{\sqrt{24\pi}} (\vec{\tau} \cdot \hat{r} \hat{r} - 3\hat{r}) \chi_{\pm} = -\sqrt{\frac{2}{3}} \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(1)} + \sqrt{\frac{1}{3}} \vec{\mathcal{Y}}_{\frac{1}{2}^+ \pm \frac{1}{2}}^{(2)}.\end{aligned}$$

They correspond exactly to $\mathcal{K}_{\frac{1}{2}^+ k_3}^{(2)}$ and $\mathcal{K}_{\frac{1}{2}^+ k_3}^{(3)}$ of Sec. 5.1, respectively.

Near the origin, the equations of motion become, asymptotically,

$$\begin{aligned}\varphi'' + \frac{2}{r}\varphi' &= 0, \\ \varphi_1^{*''} + \frac{2}{r}\varphi_1^{*'} - \frac{4}{r^2}\varphi_1^* &= -\frac{2\sqrt{2}}{r^2}\varphi_2^*, \\ \varphi_2^{*''} + \frac{2}{r}\varphi_2^{*'} - \frac{2}{r^2}\varphi_2^* &= -\frac{2\sqrt{2}}{r^2}\varphi_1^*.\end{aligned}\tag{6.53}$$

They imply that we have three independent sets of solutions:

$$\begin{aligned}\text{(i)} \quad & \begin{cases} \varphi(r) = \varphi(0) + O(r^2), \\ \varphi_i^*(r) = O(r^2), \quad (i = 1, 2) \end{cases} \\ \text{(ii)} \quad & \begin{cases} \varphi(r) = O(r^2), \\ \varphi_i^*(r) = \varphi_{bi}^*(0) + O(r^2), \quad (i = 1, 2) \end{cases} \\ \text{(iii)} \quad & \begin{cases} \varphi(r) = O(r^4), \\ \varphi_i^*(r) = \frac{1}{2}\varphi_{ci}^{*''}(0)r^2 + O(r^4), \quad (i = 1, 2) \end{cases}\end{aligned}\tag{6.54}$$

with $\sqrt{2}\varphi_1^*(0) = \varphi_2^*(0)$ for the solution set (ii) and $\varphi_1^{*''}(0) = -\sqrt{2}\varphi_2^{*''}(0)$ for the set (iii). For sufficiently large $r (\gg 1/m_P)$, the three equations decouple from each other: for example,

$$\varphi'' + \frac{2}{r}\varphi' + (\omega^2 - m_{P^*}^2)\varphi = 0.\tag{6.55}$$

Thus, the bound-state solutions ($\omega < m_P$) are

$$\begin{aligned}\varphi(r) &= \alpha \frac{e^{-r\sqrt{m_P^2 - \omega^2}}}{r}, \\ \varphi_1^*(r) &= \alpha_1 \frac{e^{-r\sqrt{m_{P^*}^2 - \omega^2}}}{r}, \\ \varphi_2^*(r) &= \alpha_2 \frac{e^{-r\sqrt{m_{P^*}^2 - \omega^2}}}{r},\end{aligned}\tag{6.56}$$

with three constants α , α_1 and α_2 .

The lowest-energy bound states are found by solving numerically the equations of motion (6.51). The results are shown in Fig. 6.1 and Table 6.1. Figure 6.1 shows the radial functions $\varphi(r)$ and $\varphi_1^*(r)$ for the D and D^* mesons (solid curve) and the B and B^* mesons (dashed curves). The radial function $\varphi_2^*(r)$ – which is hardly distinguishable from $\sqrt{2}\varphi_1^*(r)$ – is not shown there. By comparing the two cases, one can easily check that as the meson mass increases, (1) the radial function becomes more sharply peaked at the origin and (2) the role of the vector mesons becomes as important as that of the pseudoscalar mesons. Note that the radial function $\varphi_1^*(r)$ becomes comparable to $\varphi(r)$

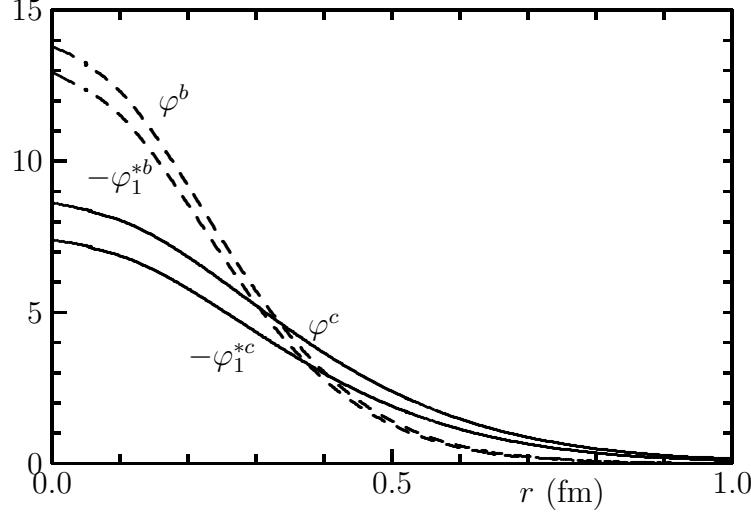


Figure 6.1 : $\varphi(r)$ and $\varphi_1^*(r)$ for $Q=c$ (solid) and b (dashed).
 $\varphi_2^*(r)$ is nearly equal to $\sqrt{2}\varphi_1^*(r)$ for both cases.

[see also the ratio $\varphi_1^*(0)/\varphi(0)$]. This can be understood as follows: due to their heavy masses, the heavy mesons are localized in the region $r \lesssim 1/m_P$, where

$$\begin{aligned} [a_1(r) + a_2(r)] &\sim [-a_1(r)] \sim F'(0) + O(r^2), \\ v(r) &\sim \frac{1}{r} - \frac{1}{4}F'^2(0)r + \dots, \end{aligned} \quad (6.57)$$

so that the equation of motion for $(\varphi_1^* - \frac{1}{\sqrt{2}}\varphi_2^*)$ is completely decoupled from those for φ and $(\varphi_1^* + \sqrt{2}\varphi_2^*)$. Note also that in Sec. 5.1 the $\mathcal{K}^{(3)}$ component is completely decoupled from the rest; *i.e.*, from the $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$ components.

The wavefunctions of Sec. 5.1 obtained in the heavy mass limit ($m_P, m_{P^*} \rightarrow \infty$), can be expressed in the same convention as

$$\begin{aligned} P &\sim \frac{1}{2} \frac{1}{\sqrt{2m_P}} f(r) \mathcal{Y}_{1/2+\pm 1/2}, \\ \vec{P}^* &\sim -\frac{1}{2} \frac{1}{\sqrt{2m_P}} f(r) (\vec{\mathcal{Y}}_{1/2+\pm 1/2}^{(1)} + \sqrt{2} \vec{\mathcal{Y}}_{1/2+\pm 1/2}^{(2)}), \end{aligned} \quad (6.58)$$

where the radial function $f(r)$, normalized to $\int r^2 dr |f|^2 = 1$, is strongly peaked at the origin. It implies that

$$\varphi(r) = -\varphi_1^*(r) = -\frac{1}{\sqrt{2}}\varphi_2^*(r) \sim \frac{1}{2} \frac{1}{\sqrt{2m_{\Phi^*}}} f(r). \quad (6.58a)$$

These radial functions satisfy the normalization condition of Eq. (6.52) in the leading order in $1/m_Q$; *viz.*,

$$2\omega_B \int_0^\infty r^2 dr (|\varphi|^2 + |\varphi_1^*|^2 + |\varphi_2^*|^2) = 1. \quad (6.58b)$$

Table 6.1 : Input parameters and the numerical results.

Q	$f_\pi^a)$	$e^c)$	$m_\Phi^a)$	$m_{\Phi^*}^a)$	$f_Q^a)$	$g_Q^c)$	$\omega_B^a)$	b.e. $^b)$	$c^c)$	$\varphi_1^*(0)/\varphi(0)$
c	64.5	5.45	1872	2010	-3016	-0.75	1481	494	0.10	-0.828
b	64.5	5.45	5275	5325	-7988	-0.75	4722	590	0.04	-0.932

$^a)$ in MeV unit, $^b)$ binding energy ($\equiv \overline{m}_P - \omega$) in MeV unit,

$^c)$ dimensionless quantities.

It is interesting to note that the pseudoscalar meson and the three vector mesons contribute equally to the bound state.

We have listed In Table 6.1 the numerical results on the lowest bound states together with the input parameters. The $SU(2)$ parameters, f_π and e , are fit to the nucleon and Δ masses and, as for the heavy meson coupling constants, the nonrelativistic quark model prediction, $g_Q = -0.75$, and the heavy quark symmetric relation, $f_Q = 2m_P g_Q$ are used. Comparing the numerical results given in Table 6.1 with the binding energy $\varepsilon_{bs} = -\frac{3}{2}gF'(0)$ of Sec. 5.1. which gives ~ 800 MeV with the same input parameters, one can see that the $1/m_Q$ corrections amount to ~ 200 MeV in the bottom sector and ~ 300 MeV in the charm sector.

The collective quantization of the soliton-heavy-meson bound state leads to the same heavy-baryon states of Eq. (5.37), so we can use the same mass formula, Eq. (6.40). Here, the hyperfine constant c is found to be^{#25}

$$\begin{aligned}
c = \int_0^\infty r^2 dr \Big\{ & 2\omega_B \left[(|\varphi|^2 - \frac{1}{3}|\varphi_1^*|^2 - \frac{2}{3}|\varphi_2^*|^2) - \frac{4}{3}\cos^2(F/2)(|\varphi|^2 - |\varphi_1^*|^2) \right] \\
& - \frac{2f_Q}{3M_{\Phi^*}} \sin F \left[\varphi^\dagger \left(\varphi_1^{*\prime} + \frac{2}{r}\varphi_1^* \right) + \left(\varphi_1^{*\prime\dagger} + \frac{2}{r}\varphi_1^{*\dagger} \right) \varphi - \frac{\sqrt{2}}{r} \sin^2 \frac{F}{2} \left(\varphi^\dagger \varphi_2^* + \varphi_2^{*\dagger} \varphi \right) \right] \\
& - \frac{1}{3}g_Q \left[F'|\varphi_2^*|^2 + \frac{\sqrt{2}}{r} \sin F (4\cos \frac{F}{2} - 1)(\varphi_1^{*\dagger} \varphi_2^* + \varphi_2^{*\dagger} \varphi_1^*) \right] \Big\},
\end{aligned} \tag{6.59}$$

where the terms next-to-leading order in $1/m_Q$ have been kept. One sees that the hyperfine constant c is of order $1/m_Q$. This is a consistency check. The leading order terms proportional to ω_B vanish identically when the radial functions of Eq. (6.58) are used.

With the bound state energy ω_B and the hyperfine constant c given in Table 6.1, the mass formula (6.40) predicts the heavy baryon masses as shown in Table 6.2 (Result I). Result II is obtained by taking the two coupling constants as free parameters. To fit the experimental masses of Λ_c and Σ_c , one should have $f_Q/2M_{D^*} = -0.88$ and $g_Q = -1.00$, which indicates that the heavy quark symmetric relation $f_Q = 2m_P g_Q$ is broken in the charm sector and shows about 25% difference with the estimate of g made in Sec. 2.2.

^{#25} An error committed in Ref. 37 is corrected here.

Table 6.2 : Numerical results on the heavy baryon masses.

Q		$f_Q/2M_{\Phi^*}$	g_Q	$\omega_B^{a)}$	c	$M_{\Lambda_Q}^{a)}$	$M_{\Sigma_Q}^{a)}$	$M_{\Sigma_Q^*}^{a)}$
	exp.					2285 ^{b)}	2453 ^{b)}	2530 ^{c)}
c	I	-0.75	-0.75	1481	0.10	2427	2596	2625
	II	-0.88	-1.00	1339	0.10	2285	2454	2483
b	exp. ^{b)}					5641 ^{b)}	—	—
	I	-0.75	-0.75	4722	0.04	5664	5849	5860

^{a)} in MeV unit, ^{b)} Particle Data Group,¹⁰ ^{c)} Ref. 11.

7. Conclusions

In this review, we have described how chiral symmetry of light quarks and heavy-quark symmetry of heavy quarks can be combined in a skyrmion description of the heavy baryons. That heavy baryons can be described as skyrmions may appear to some readers as surprising as one would naively think that the soliton idea would be inappropriate for a system where one or more heavy quarks are “bound” to the soliton which in reality with $N_c = 3$ is not so heavy compared with the heavy meson. This would seem to give an absurd picture of the system as “a tail wagging a dog.” So how does this work?

There are two points to this paradox. The first is that the picture is built with both the number of colors N_c and the heavy quark mass going to infinity. In this limit, both the soliton and the meson are infinitely heavy and so on top of each other. The second point is that in actual fact for $N_c \neq \infty$, the heavy meson is “wrapped” by the soliton in contrast to the monopole-scalar field system where the scalar field is “pierced” by the soliton.

We have seen that the two approaches, the “top-down” method that comes down from heavy-quark limit and the “bottom-up” approach which goes up from chiral symmetry limit, give the same description. How they are related is shown in this review but the deep reason behind this relation, if it exists, is not known.

In the latter approach, the heavy-quark limit emerges in an intriguing way through the vanishing of a nonabelian Berry potential, with a finite non-vanishing Berry potential describing in an approximate but rather accurate way the hyperfine splitting of heavy baryons. The interesting question as to how to compute corrections to hyperfine splittings that are not included in the Berry potentials but would be needed for not-so-heavy baryons (such as strange and charmed hyperons) has not been addressed in this paper and remains an open question. This may be related to non-adiabatic corrections to Berry potentials. There is also a problem as to at what quark mass the Wess-Zumino term vanishes in the “bottom-up” approach. We have seen that the ω -exchange term in

the CK skyrmion is attractive for K^- and repulsive for K^+ but when the Wess-Zumino term disappears, the role of the ω field is not determined. So what happens to the ω field as the Wess-Zumino term is about to disappear?

Many applications of the skyrmion description to heavy-baryon phenomenology still remain to be worked out although some are just appearing in the literature. What is crucially needed however is more experiments in the field.

Acknowledgement

We are grateful for discussions with M. A. Nowak, N. N. Scoccola and I. Zahed. This work was supported in part by the Korea Science and Engineering Foundation through the Center for Theoretical Physics of Seoul National University and by the Chungnam National University Research and Scholarship Fund. One of us (Y.O.) was supported in part by the National Science Council of ROC under grant #NSC84-2811-M002-036.

References

- [1] M. B. Voloshin and M. A. Shifman, *Yad. Fiz.* **45**, 463 (1987), **47**, 801 (1988), [*Sov. J. Nucl. Phys.* **45**, 292 (1987); **47**, 511 (1988)].
- [2] S. Nussinov and W. Wetzel, *Phys. Rev.* **D36**, 130 (1987); E. Eichten, *Nucl. Phys.* **B (Suppl.) 4C**, 170 (1988); G. P. Lepage and B. A. Thacker, *ibid.* **4**, 199 (1988); H. D. Politzer and M. B. Wise, *Phys. Lett.* **B206**, 681 (1988); **208**, 504 (1988).
- [3] N. Isgur and M. B. Wise, *Phys. Lett.* **B232**, 113 (1989); **237**, 527 (1990).
- [4] H. Georgi, *Phys. Lett.* **B240**, 447 (1990).
- [5] H. Georgi, in *Proc. of the Theoretical Advanced Study Institute*, eds. R. K. Ellis *et al.* (World Scientific, Singapore, 1992) and references therein.
- [6] M. B. Wise, in *Proc. of the Sixth Lake Louise Winter Institute*, eds. B.A. Campbell *et al.* (World Scientific, Singapore, 1991).
- [7] B. Grinstein, *Annu. Rev. Nucl. Part. Sci.* **42**, 10 (1992).
- [8] M. Neubert, *Phys. Rep.* **245**, 259 (1994).
- [9] N. Isgur and M. B. Wise, *Phys. Rev. Lett.* **66**, 1130 (1991).
- [10] Particle Data Group, K. Hikasa *et al.*, *Phys. Rev.* **D45**, Part II (1993).
- [11] SKAT Collaboration, V. V. Ammosov *et al.*, *Pis'ma Zh. Eksp. Teor. Fiz.* **58**, 241 (1993) [*JETP Lett.* **58**, 247 (1993)].
- [12] C. G. Callan and I. Klebanov, *Nucl. Phys.* **B262**, 365 (1985); C. G. Callan, K. Hornbostel and I. Klebanov, *Phys. Lett.* **B202**, 269 (1988).
- [13] D. B. Kaplan and I. Klebanov, *Nucl. Phys.* **B335**, 45 (1990).
- [14] N. N. Scoccola, H. Nadeau, M. A. Nowak and M. Rho, *Phys. Lett.* **B201**, 425 (1988) (E) **220**, 658 (1989); J.-P. Blaizot, M. Rho and N. N. Scoccola, *ibid.* **209**, 27 (1988); N. N. Scoccola and A. Wirzba, *ibid.* **258**, 451 (1991).
- [15] U. Blom, K. Dannbom and D. O. Riska, *Nucl. Phys.* **A493**, 384 (1989); J. Kunz and P. J. Mulders, *Phys. Lett.* **B231**, 335 (1989); *Phys. Rev.* **D41**, 1578 (1990); D.-P. Min, Y. S. Koh, Y. Oh and H. K. Lee, *Nucl. Phys.* **A530**, 698 (1991); J. Kunz, P. J. Mulders and G. A. Miller, *Phys. Lett.* **B255**, 11 (1991).

- [16] C. Gobbi, D. O. Riska and N. N. Scoccola, *Nucl. Phys.* **A544**, 671 (1992); G. L. Thomas, N. N. Scoccola and A. Wirzba, *ibid.* **575**, 623 (1994).
- [17] Y. Kondo, S. Saito and T. Otofujii, *Phys. Lett.* **B236**, 1 (1990); **256**, 316 (1991); K. Dannbom and D. O. Riska, *Nucl. Phys.* **A548**, 669 (1992).
- [18] N. N. Scoccola, D. P. Min, H. Nadeau and M. Rho, *Nucl. Phys.* **A505**, 497 (1989).
- [19] M. Rho, D. O. Riska and N. N. Scoccola, *Phys. Lett.* **B251**, 597 (1990).
- [20] D. O. Riska and N. N. Scoccola, *Phys. Lett.* **B265**, 188 (1991); M. Rho, D. O. Riska and N. N. Scoccola, *Z. Phys.* **A341**, 343 (1992).
- [21] Y. Oh, D.-P. Min, M. Rho and N. N. Scoccola, *Nucl. Phys.* **A535**, 493 (1991).
- [22] See also M. Oka, *Phys. Lett.* **B205**, 1 (1988).
- [23] Y. Oh, Ph. D. Thesis, Seoul National University, 1993 (unpublished).
- [24] A. De Rujula, H. Georgi and S. L. Glashow, *Phys. Rev.* **D12**, 147 (1975).
- [25] E. V. Shuryak, *Phys. Lett.* **B93**, 134 (1980); D. Izatt, C. Detar and M. Stephenson, *Nucl. Phys.* **B199**, 269 (1982).
- [26] G. Pari, B. Schwesinger and H. Walliser, *Phys. Lett.* **B255**, 1 (1991); G. Pari and N. N. Scoccola, *ibid.* **296**, 391 (1992).
- [27] M. Björnberg, K. Dannbom, D. O. Riska and N. N. Scoccola, *Nucl. Phys.* **A539**, 662 (1992); M. Björnberg and D. O. Riska, *ibid.* **549**, 537 (1992).
- [28] E. Jenkins, A. V. Manohar and M. B. Wise, *Nucl. Phys.* **B396**, 27 (1993).
- [29] Z. Guralnik, M. Luke and A. V. Manohar, *Nucl. Phys.* **B390**, 474 (1993).
- [30] D. P. Min, Y. Oh, B.-Y. Park and M. Rho, SNU Report SNUTP-92/78 (unpublished).
- [31] E. Jenkins and A. V. Manohar, *Phys. Lett.* **B294**, 273 (1992).
- [32] M. A. Nowak, M. Rho and I. Zahed, *Phys. Lett.* **B303**, 130 (1993).
- [33] K. S. Gupta, M. A. Momen, J. Schechter and A. Subbaraman, *Phys. Rev.* **D47**, 4835 (1993).
- [34] H. K. Lee, M. A. Nowak, M. Rho and I. Zahed, *Ann. Phys. (N.Y.)* **227**, 175 (1993); H. K. Lee and M. Rho, *Phys. Rev.* **D48**, 2329 (1993).
- [35] G. Burdman and J. F. Donoghue, *Phys. Lett.* **B280**, 287 (1992).
- [36] M. B. Wise, *Phys. Rev.* **D45**, 2188 (1992).
- [37] Y. Oh, B.-Y. Park and D.-P. Min, *Phys. Rev.* **D49**, 4649 (1994).
- [38] J. D. Bjorken, in *La Thuile Rencontres*, invited talk at Les Rencontres de Physique de la Vallée d'Aoste, La Thuile, Italy, SLAC Report No. SLAC-PUB-5278.
- [39] A. F. Falk, *Nucl. Phys.* **B378**, 79 (1992).
- [40] E. Witten, *Nucl. Phys.* **B223**, 422 (1983); **233**, 433 (1983).
- [41] J. Wess and B. Zumino, *Phys. Lett.* **B37**, 95 (1971).
- [42] A. Manohar and H. Georgi, *Nucl. Phys.* **B234**, 189 (1984).
- [43] H. Georgi, “*Weak Interactions and Modern Particle Theory*” (Benjamin/Cummings, Menlo Park, 1984).
- [44] T. M. Yan, H.-Y. Cheng, C.-Y. Cheung, G.-L. Lin, Y. C. Lin and H.-L. Yu, *Phys. Rev.* **D46**, 1148 (1992).

- [45] CUSB Collaboration, K. Han *et al.*, *Phys. Rev. Lett.* **55**, 36 (1985).
- [46] S. Weinberg, *Phys. Rev. Lett.* **65**, 1181 (1990); **67**, 3473 (1991); S. Peris, *Phys. Lett.* **B268**, 415 (1991).
- [47] ACCMOR Collaboration, S. Barlag *et al.*, *Phys. Lett.* **B278**, 480 (1992).
- [48] CLEO Collaboration, S. Butler *et al.*, *Phys. Rev. Lett.* **69**, 2041 (1992).
- [49] T. H. R. Skyrme, *Proc. Roy. Soc. (London)* **A260**, 127 (1961); **262**, 237 (1961); *Nucl. Phys.* **31**, 556 (1962); *Int. Jour. Mod. Phys.* **A3**, 2745 (1988).
- [50] A. Chodos, E. Hadjimichael and C. Tze, eds. “*Solitons in Nuclear and Elementary Particle Physics*”, Proc. of the Lewes Workshop, 1984 (World Scientific, Singapore, 1984).
- [51] I. Zahed and G. E. Brown, *Phys. Rep.* **142**, 1 (1986).
- [52] K.-F. Liu, ed. “*Chiral Solitons*” (World Scientific, Singapore, 1987).
- [53] U.-G. Meissner, *Phys. Rep.* **161**, 213 (1988).
- [54] B. Schwesinger, H. Weigel, G. Holzwarth and A. Hayashi, *Phys. Rep.* **173**, 173 (1989).
- [55] E. M. Nyman and D. O. Riska, *Rep. Prog. Phys.* **53**, 1137 (1990).
- [56] M. Rho, *Phys. Rep.* **240**, 1 (1994).
- [57] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* **158**, 142 (1984); *Nucl. Phys.* **B250**, 465 (1985).
- [58] M. A. Nowak, M. Rho and I. Zahed, *Phys. Rev.* **D48**, 4370 (1993); J. Soto and R. Tzani, *Phys. Lett.* **B297**, 358 (1992).
- [59] J. Schwinger, *Phys. Lett.* **B24**, 473 (1967); J. Wess and B. Zumino, *Phys. Rev.* **163**, 1727 (1967); S. Gasiorowicz and D. Geffen, *Rev. Mod. Phys.* **41**, 473 (1969).
- [60] M. Bando, K. Kugo, S. Uehara, K. Yamawaki and T. Yanagida, *Phys. Rev. Lett.* **54**, 1215 (1985).
- [61] For a review see *e.g.*, M. Bando, T. Kugo and K. Yamawaki, *Phys. Rep.* **164**, 217 (1988) and references therein.
- [62] E. G. C. Stükelberg, *Helv. Acta. Phys.* **14**, 51 (1941); K. Yamawaki, *Phys. Rev.* **D35**, 412 (1987).
- [63] J. Schechter, *Phys. Rev.* **D34**, 868 (1986); P. Jain, R. Johnson, U.-G. Meissner, N. W. Park and J. Schechter, *ibid.* **37**, 3252 (1988); U.-G. Meissner, N. Kaiser, H. Weigel and J. Schechter, *ibid.* **39**, 1956 (1989); M. Wakamatsu, *Ann. Phys. (N.Y.)* **193**, 287 (1989).
- [64] K. Kawarabayashi and M. Suzuki, *Phys. Rev. Lett.* **15**, 255 (1966); Riazuddin and Fayyazuddin, *Phys. Rev.* **147**, 1071 (1966).
- [65] Y. Igarashi, M. Johmura, A. Kobayashi, H. Otsu, T. Sato and S. Sawada, *Nucl. Phys.* **B259**, 721 (1985); Y. Igarashi, A. Kobayashi, H. Otsu and S. Sawada, *Prog. Theor. Phys.* **78**, 358 (1987); M. Abud, G. Maiella, F. Nicodemi, R. Pettorino and K. Yoshida, *Phys. Lett.* **B159**, 155 (1985).
- [66] T. Fujiwara, T. Kugo, H. Terao, S. Uehara and K. Yamawaki, *Prog. Theor. Phys.* **73**, 926 (1985).

- [67] U.-G. Meissner, N. Kaiser, A. Wirzba and W. Weise, *Phys. Rev. Lett.* **57**, 1676 (1986).
- [68] M. Bando, T. Kugo and K. Yamawaki, *Nucl. Phys.* **B259**, 493 (1985).
- [69] A. Bramon, A. Grau and G. Pancheri, “Effective chiral Lagrangians with an $SU(3)$ -broken vector-meson sector,” UAB-FT-335/94, hep-ph/9411269.
- [70] P. Jain, R. Johnson, N. W. Park, J. Schechter and H. Weigel, *Phys. Rev.* **D40**, 855 (1989).
- [71] J. M. Flynn and N. Isgur, *J. Phys.* **G18**, 1627 (1992).
- [72] G. S. Adkins, C. R. Nappi and E. Witten, *Nucl. Phys.* **B228**, 552 (1983).
- [73] A. D. Jackson and M. Rho, *Phys. Rev. Lett.* **51**, 751 (1983).
- [74] M. Rho, *Mod. Phys. Lett.* **A6**, 2087 (1991); *Act. Phys. Pol.* **B22**, 1001 (1991).
- [75] A. Shapere and F. Wilczek, ed., “*Geometric phases in physics*” (World Scientific, Singapore, 1989).
- [76] A. Bohr and B. Mottelson, *Nuclear Structure, Vol. 2* (Benjamin, New York, 1969).
- [77] N. N. Scoccola and A. Wirzba, in Ref. 14.
- [78] Y. Oh, B.-Y. Park and D.-P. Min, *Phys. Rev.* **D50**, 3350 (1994).
- [79] E. Jenkins, A. V. Manohar and M. B. Wise, *Nucl. Phys.* **B396**, 38 (1993).
- [80] A. Momen, J. Schechter and A. Subbaraman, *Phys. Rev.* **D49**, 5970 (1994).
- [81] Y. Oh, B.-Y. Park and D.-P. Min, *Phys. Lett.* **B331**, 362 (1994).
- [82] J. Schechter and A. Subbaraman, *Phys. Rev.* **D48**, 332 (1993).
- [83] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gotto, F. Feruglio and G. Nardulli, *Phys. Lett.* **B292**, 371 (1992); **299**, 139 (1993).
- [84] R. Casalbuoni, A. Deandrea, N. Di Bartolomeo, R. Gotto, F. Feruglio and G. Nardulli, *Phys. Lett.* **B312**, 315 (1993).
- [85] Ö. Kaymakçalan and J. Schechter, *Phys. Rev.* **D31**, 1109 (1985).
- [86] U.-G. Meissner, N. Kaiser and W. Weise, *Nucl. Phys.* **A466**, 685 (1987).
- [87] G. S. Adkins and C. R. Nappi, *Phys. Lett.* **B137**, 251 (1984).
- [88] Fermilab E653 Collaboration, K. Kodama *et al.*, *Phys. Lett.* **B286**, 187 (1992).